Liquidity Premia in Dynamic Bargaining Markets

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Abstract

This paper develops a search-theoretic model of the cross-sectional distribution of asset returns, abstracting from risk premia and focusing exclusively on liquidity. In contrast with much of the transaction-cost literature, it is not assumed that different assets carry different exogenously specified trading costs. Instead, different expected returns, due to liquidity, are explained by the cross-sectional variation in tradeable shares. The qualitative predictions of the model are consistent with much of the empirical evidence.

Keywords: Liquidity premia, Search

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1 Introduction

Why do different assets earn different expected returns? One fundamental reason is that they may bear different risks. Many empirical studies, however, suggest that risk characteristics cannot explain all variation in expected returns. After controlling for risk premia, expected returns appear to be positively related to bid-ask spreads, and negatively related to turnover and dollar trading volume. These patterns suggest that returns are related to liquidity, broadly defined as the ease of buying and selling. Liquidity is reflected in small trading costs, measured for instance by the bid-ask spread, and associated with the opportunity to buy or sell large quantities in a short time, at a similar price. These properties may be proxied by turnover or trading volume.

This paper provides a dynamic asset pricing model in which cross-sectional variation in asset returns is exclusively due to liquidity differences. In our model, investors cannot trade instantly in multilateral Walrasian market. Instead, trade is bilateral: investors have to search for each others, meet in pairs, and bargain over prices. In this environment, a more liquid asset is defined as one with smaller trading delays: buyers and sellers of that asset are more likely to be found in a short time. This search framework applies most directly to over-the-counter markets such as the Treasury market, the corporate-bond market, or markets for financial derivatives. More generally, it applies to trades that are not arranged in a centralized market, such as block trades in the New York Stock Exchange (NYSE) upstairs market. Lastly, the search friction is likely to have an impact on asset prices even in markets where security dealers provide immediacy to outside investors. Indeed, the search friction determines investors' outside option when they trade with dealers. In addition, dealers might have to search for end investors in order to unload their inventories, and would charge the associated search cost to their customers. Lastly, in some markets, such as the corporate-bond market, dealers typically act as a brokers and search for counterparties on the behalves of their customers.

In the present model, many different assets are traded. Investors allocate their fixed budgets of search efforts to the various assets. They recognize that the value of searching for a particular asset is related to the likelihood of finding a counterparty for that asset in a short time. The first-order condition of the associated search optimization problem is key to the model's implications, as it reflects how the likelihood of finding an asset is priced in equilibrium. Specifically, in equilibrium, investors are indifferent between searching for alternative traded assets, under natural technical conditions. This indifference property gives rise to a distribution of "liquidity premia." Namely, an asset that is easier to find is sold at a higher price.

In traditional Walrasian asset-pricing models with liquidity effects, such as those of Amihud and Mendelson [1986], Constantinides [1986], Heaton and Lucas [1996], Vayanos [1998], and Huang [2003], assets can be bought and sold instantly, but differ by an exogenously given transaction cost. A more liquid asset is defined as one with a smaller transaction cost. In these models, cross-sectional variation in asset returns is explained by exogenously specified differences in transaction costs. In contrast, this paper explains cross-sectional variation in asset returns without relying on an exogenously specified cross-sectional variation in transaction costs. Although, in the model proposed here, the search technology is the same for all assets, heterogeneous bid-ask spreads arise endogenously. Cross-sectional variation in asset returns is explained by the distribution of tradeable shares.

This paper extends the one-asset models of Duffie, Gârleanu, and Pedersen [2005, 2007] by allowing investors to trade many assets. The present cross-sectional analysis could not have been conducted in the one-asset model, which examines the impact of liquidity on asset prices only by comparative statics. In particular, in the one-asset model, an increase in the quantity of tradeable shares results in a positive shift of the supply curve, and thus decreases the price of the asset. In the multiple-assets model, one can keep the aggregate number of tradeable shares constant, and study an equilibrium in which some assets have more tradeable shares than others. This isolates a liquidity effect: under natural conditions, an asset with more tradeable shares is easier to find, and has a higher price. More broadly, this effect goes against a risk prediction,[1] as well as the prediction of Hong, Scheinkman, and Xiong [2006] based on differences of opinions.

In the last part of the paper we extend our analysis to a general, well-behaved, matching technology. We show that an equilibrium exists and, as in our baseline model, that cross-sectional returns are negatively related to turnover and positively related to bid-ask spread. We also show that some predictions of the model depend on the curvature of the

[1] For instance, in a standard Constant Absolute Risk Aversion model with one asset, an increase in supply decreases the price only because it increases the supply of risk, i.e. the number of tradeable shares times the risk per share.

matching function. For instance, consider standard liquidity proxies such as turnover or (the negative of) the bid-ask spread. Then we find that, in equilibrium, liquidity is positively related to the number of tradeable shares if the matching function has increasing returns to scale, and negatively related to the number of tradeable shares if the matching function has decreasing returns. Hence, according to the model, the sign of the cross-sectional relationship between liquidity and tradeable shares allows to empirically distinguish between increasing and decreasing returns in matching.

Search-theoretic approaches to liquidity have been explored in the monetary literature, following Kiyotaki and Wright [1989]. Most notably, Wallace [2000] focuses on the relative liquidity of intrinsically worthless assets (currency) and assets earning a positive dividend (bonds). The model presented here has no room for currency, and focuses on assets with relatively homogeneous characteristics. The recent work of Lagos [2006] studies liquidity difference between stocks and bonds in a search model designed to nest the consumption-based asset pricing model of Mehra and Prescott [1985]. Lastly, the present paper is closely related to the independent work of Vayanos and Wang [2007]. In order to study liquidity difference between on-the-run and off-the-run bonds, they provide a two-asset extension of Duffie, Gârleanu, and Pedersen [2005]. They analyze the impact of investor heterogeneity on the concentration of liquidity across markets, and focus most of their analysis on welfare. In the present paper, by contrast, we analyze the impact of asset heterogeneity, extending our result to a general matching function, and focus most of the analysis on pricing and measurement.

The remainder of the paper is organized as follows. Section 2 describes the setup, Section 3 defines, calculates, and analyze an equilibrium where buyers search for all assets. Lastly, Section 4 extends the results to a general matching technology and discusses equilibria where buyers search only for a subset of assets. The Appendix collects all the proofs.

2 Model Setup

This section presents the basic model, in which investors cannot buy and sell assets instantly. Rather, they allocate search resources to asset-specific "trading specialists," who search for counterparties. When two investors meet, they bargain over the terms of trade. (The specialists could bargain on their behalves.)

2.1 Information and Preferences

Time is treated continuously, and runs forever. A probability space $(\Omega, \mathcal{F}, P)$ is fixed, as well as a filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions (Protter [1990]). There are many assets $k \in \{1, \ldots, K\}$ in positive supply. Asset k has a measure $s_k \in (0, 1)$ of tradeable shares, and every share of an asset pays the same dividend rate $\delta > 0$.

The economy is populated by a unit-mass continuum of infinitely-lived and risk-neutral investors who discount the future at the constant rate $r > 0$. An investor enjoys the consumption of a non-storable numéraire good called "cash," with a marginal utility normalized to 1. In order to make side payments, investors are endowed with a technology that instantly produces cash, at unit marginal cost.[2]

An investor has either a high-valuation or a low-valuation for holding assets. When he has a high valuation and holds asset $k \in \{1, \ldots, K\}$, he enjoys the (per unit) utility flow δ. With a low valuation, he enjoys a utility flow $\delta - x$, for some holding cost $x > 0$.[3] Investors switch randomly, and pair-wise independently, from a low valuation to a high valuation with intensity[4] γ_u, and from a high valuation to a low valuation with intensity γ_d.

2.2 Asset Holdings

An investor is permitted to hold either zero or one share of some asset,[5] and can choose which asset to hold. We let $s \equiv (s_1, s_2, \ldots, s_K)$ denote the distribution of tradeable shares.

[2]In other words, negative consumption of cash is allowed. Equivalently, one could assume that investors borrow and save cash in some "bank account," at the exogenously given interest rate $\bar{r} = r$, and subject to an appropriate transversality condition.

[3]Duffie, Gârleanu, and Pedersen [2005] and Vayanos and Weill [2007] provide a formal model of the holding cost. They assume that risk-averse investors receive some non-tradable endowment stream which is sometimes highly correlated with the traded asset. In a first-order Taylor expansion of an investor's continuation utility, x represents the cost of holding an asset when it has a high correlation with the endowment.

One could also view the holding cost as the intensity of an investor's need for cash, when he is borrowing constrained and cannot borrow against the full value of his asset holding. Suppose that, if the asset is worth p_k, an investor can only borrow $p_k - h$, for some "haircut" h. If the shadow value of relaxing the borrowing constraint is ϕ, then the holding cost is $x = \phi h$.

[4]For instance, if the investor's valuation is low, the distribution of the next switching time to high is exponential with parameter γ_u. The successive switching times are independent.

[5]Because he has linear utility over dividend, an investor finds it optimal to hold either the minimum quantity of zero share, or the maximum quantity of one share. Normalizing the maximum holding to be one share is without loss of generality, in the following sense: the results would remain unchanged if one assumes a maximum holding of N shares, and redefine the dividend rate to be δ/N.

We also assume that

$$\sum_{k=1}^{K} s_k \equiv S < \frac{\gamma_u}{\gamma_u + \gamma_d}, \tag{1}$$

which means that the total supply S of assets is less than the steady-state measure of high-valuation investors.[6] Given that investors can hold at most one unit of some asset, equation (1) implies that, in a multilateral Walrasian market, the "marginal investor" has a high valuation. Therefore, in a Walrasian market, all assets have the same equilibrium price δ/r.

An investor's type is made up of her marginal utility (high h, or low ℓ), and her ownership status, for each asset type $k \in \{1, \ldots, K\}$ (owner ok, or nonowner n). Hence, the set of investor types is

$$I = \{hn, \ell n, ho1, \ldots, hoK, \ell o1, \ldots, \ell oK\}. \tag{2}$$

In anticipation of their equilibrium behavior, high-utility non owner (hn) are named "buyers," and low-utility owners of asset k (ℓok) are named "sellers of asset k." For each $i \in I$, μ_i denotes the fraction of investors of type i, and, given the asset fundamentals and the trading environment (to be defined), V_i denotes the continuation utility of an investor of type i.

2.3 Matching Technology

At any point in time, each investor is endowed with a mass $\bar{\nu}$ of "trading specialists" who search for specific trading counterparties, in a sense that is now to be described.

A trading specialist of type $(i, j) \in I^2$ works for an investor of type i, and specializes in contacting specialists working for investors of type j. Thus, contacts that could result in a trade occur only between specialists of types (i, j) and (j, i). An investor of type i maintains on her "trading staff" a quantity ν_{ij} of specialists of type (i, j), subject to the resource constraint $\sum_{j \in I} \nu_{ij} \leq \bar{\nu}$. Thus, the mass of specialists of type (i, j) in the entire specialist population is $\mu_i \nu_{ij}$. A given specialist makes contacts with other specialists, pair-wise independently at Poisson arrival times with intensity $\lambda > 0$. Because scaling $\bar{\nu}$ and λ up and down, respectively, by the same factor has no effect, one can assume without loss of

[6] An application of the law of large numbers (Sun [2000]) implies that the steady-state measure of high-valuation investors is equal to the stationary probability $\gamma_u/(\gamma_d + \gamma_u)$ of being in a state of high valuation.

generality that $\bar{\nu} = 1$. Contacts are also pair-wise independent with the investor's valuation processes. Given a contact, because of the random-matching assumption, the probability that the contact is made with a specialist of type (i, j) is $\mu_i \nu_{ij}$. That is, conditional on making a contact, all trading specialists in the entire specialist population are "equally likely" to be contacted. The results of Duffie and Sun [2007] ensure that an independent random matching exists among our continuum of specialists, so that contacts between specialists of types (i, j) and (j, i), for $i \neq j$, occur continually at a total (almost sure) rate of

$$\mu_i \nu_{ij} \lambda \mu_j \nu_{ji} + \mu_j \nu_{ji} \lambda \mu_i \nu_{ij} = 2\lambda \mu_i \nu_{ij} \mu_j \nu_{ji}. \tag{3}$$

The first term on the left-hand side of (3) is the total rate of contacts made by all specialists of type (i, j), and received by specialists of type (j, i). Specifically, each specialist of the mass $\mu_i \nu_{ij}$ of specialists of type (i, j) makes contacts at rate λ, and such contacts are received by some specialist of type (j, i) with probability $\mu_j \nu_{ji}$. Similarly, the second term is the total rate of contact made by specialists of type (j, i) and received by specialists of type (i, j).

On advantage of using the aggregate contact rate (3) is that it arises from an explicitly specified random search process. Many authors (e.g., Pissarides [2000]) go the other way: they directly postulate that aggregate rate of contact between two searching populations of respective measure μ_a and μ_b is given by some "well behaved" matching function $M(\mu_a, \mu_b)$.[7] In Section 4, we characterize an equilibrium of the model with such a reduced-form matching function.

2.4 Discussion

An investor maintaining trading specialists can be viewed as an investment firm with separate units that trade specific securities. A typical unit trades securities of a specific industry, such as "telecom" or "entertainment," or trades securities sharing a similar payoff structure, such as stocks, fixed-income, or derivatives. Specialization in trading reflects the costs of collecting and processing information regarding the supply and demand of assets, as well as the fundamentals of the underlying cash flows. In practice, investors tend to specialize

[7] While this approach has the benefit of flexibility, one does not know whether there exists an individual search process consistent with the postulated aggregate behavior (see Lagos [2000] for a critique and an alternative approach).

in broad asset classes rather than in individual securities: specializing in too fine categories would presumably be very costly, as it is likely to create substantial under-diversification.

In light of this discussion, the assets of our model are best viewed as, say, industry portfolios. A portfolio interpreation also makes it more plausible to assume that investors can hold only one asset. Indeed, one would expect a substantial amount of idiosyncratic risk to be diversified away in such industry portfolios, so that the benefit of trade specialization may balance the cost of under-diversification.

3 A Symmetric Equilibrium

This section defines, calculates, and analyzes an equilibrium in which buyers find it optimal to search for all assets.

3.1 Equilibrium Definition and Characterization

We first analyze investors' decisions: whether or not to trade in a given encounter, and how to allocate trading specialists across types of trading encounters. Then, we describes the dynamics of the distribution of types.

3.1.1 Trade Among Investors

Trade between investors of types i and j occurs at a strictly positive rate if (a) the gain from trade from such a pair is strictly positive,[8] and (b) these two types of investors maintain trading specialists who are searching for each other, that is, if $\nu_{ij}\nu_{ji} > 0$.

In equilibrium, the gains from trade will be strictly positive when a seller of asset k, ℓok, contacts a buyer, hn. The ℓok investor will sell her asset to the hn investor, in exchange for some cash payment p_k.[9] The price arises in a simple Nash-bargaining game, as follows. The total surplus of such a transaction is

$$(V_{hok} - V_{hn}) - (V_{\ell ok} - V_{\ell n}) \equiv \Delta V_{hk} - \Delta V_{\ell k}. \tag{4}$$

We study those equilibria in which the ℓok agent receives a fixed fraction $q \in (0, 1)$ of the total surplus. This implies that the price of asset k is, in an equilibrium,

$$p_k = q\Delta V_{hk} + (1 - q)\Delta V_{\ell k}. \tag{5}$$

[8]An arbitrarily small transaction cost rules out trade when the gain is zero.

[9]A cash payment payments is a lump of consumption good, instantly produced at unit marginal cost.

The gains from trade can also be positive between a low-valuation owner ℓok and a high-valuation owner hoj. These two investors may swap assets, and one investor may simultaneously transfer cash to the other. The total surplus of a swap between a ℓok agent and a hoj agent is $V_{\ell oj} - V_{\ell ok} + V_{hok} - V_{hoj}$.

We solve for an equilibrium where hn investors search for all assets, and ℓok investors search only for an outright sale with hn investors, but do not search for swaps. Precisely, at each time, an hn investor maintains a measure $\nu_k > 0$ of trading specialists who seek to buy asset $k \in \{1, \dots, K\}$ from ℓok investors. On the other side of the search market, an ℓok investor only maintains trading specialists who search for an outright sale with hn investors. The allocation of trading specialists of hn and ℓok investors are illustrated in Figure 1.

Importantly, in the equilibrium we analyze, an ℓok investor does not search for swaps: in other words, the net utility of searching for a swap ends up strictly less than the net utility of searching for an outright sale, a condition that can be written

$$2\lambda \nu_{jk} \mu_{hoj} q (V_{\ell oj} - V_{\ell ok} + V_{hok} - V_{hoj}) < 2\lambda \nu_k \mu_{hn} q (\Delta V_{hk} - \Delta V_{\ell k}), \tag{6}$$

for all $(k, j) \in \{1, \dots, K\}^2$, and where ν_{jk} denotes the fraction of trading specialists that hoj investors allocate to the search of a swap with ℓok investors. Simplifying $2\lambda q$ from both sides, and noting that $\nu_{jk} \leq 1$, it follows that (6) will hold if

$$\mu_{hoj} (V_{\ell oj} - V_{\ell ok} + V_{hok} - V_{hoj}) < \nu_k \mu_{hn} (\Delta V_{hk} - \Delta V_{\ell k}). \tag{7}$$

for all $(k, j) \in \{1, \dots, K\}^2$. We verify this sufficient condition in the proof of Propositions 3, 4, and 6.

Definition 1. *An allocation of trading specialist is some $\nu \in \mathbb{R}_+^K$ with $\sum_{k=1}^K \nu_k \leq 1$.*

3.1.2 Bellman Equations

This paragraph characterizes the equilibrium continuation utilities V_i, $i \in I$. The Bellman equation for the continuation utility of a buyer hn is

$$rV_{hn} = \max_{\tilde{\nu}_1, \dots, \tilde{\nu}_K} \left\{ \gamma_d (V_{\ell n} - V_{hn}) + \sum_{k=1}^K 2\lambda \tilde{\nu}_k \mu_{\ell ok} \left(V_{hok} - V_{hn} - p_k \right) \right\}, \tag{8}$$

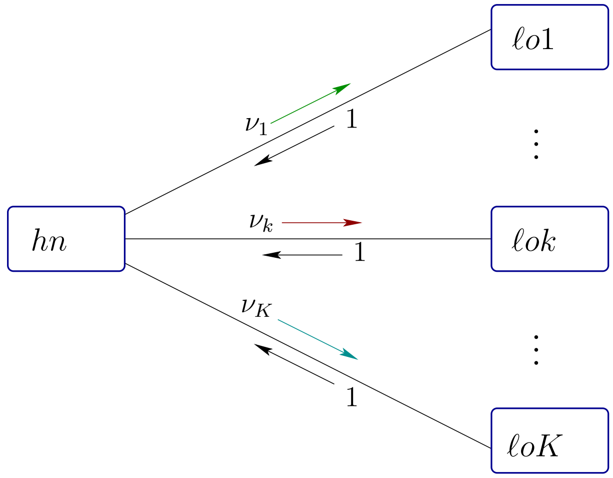

Figure 1: Allocating Search Intensity

An hn investor allocates a fraction ν_k of trading specialists to the search of ℓok investors, for all $k \in \{1, \ldots, K\}$. An ℓok investor, on the other-hand, allocates all of her specialists to the search of hn investors.

subject to $\sum_{k=1}^{K} \tilde{\nu}_k \leq 1$ and $\tilde{\nu}_k \geq 0$, for all $k \in \{1, \ldots, K\}$. Tilde notation ($\tilde{}$) is used to distinguish the specialist allocation ν_k that will prevail in equilibrium for all investors of type hn from the allocation $\tilde{\nu}_k$ that is to be chosen by an individual investor of type hn, taking others' allocation as given. The Bellman equation (8) breaks up the "flow" continuation utility rV_{hn} into two terms. The first term, $\gamma_d(V_{\ell n} - V_{hn})$, is the expected flow utility of a transition from a high to a low valuation because, with intensity γ_d, an hn investor makes a transition to the ℓn type. The second term is the expected flow utility of searching for alternative assets. Namely, with intensity $2\lambda\tilde{\nu}_k\mu_{\ell ok}$, an hn investor finds asset k, buys it at price p_k and makes a transition to type hok. Similarly, other investors' continuation utilities solve the following system of Bellman equations

$$rV_{hok} = \delta + \gamma_d(V_{\ell ok} - V_{hok}) \tag{9}$$

$$rV_{\ell ok} = \delta - x + \gamma_u(V_{hok} - V_{\ell ok}) + 2\lambda\nu_k\mu_{hn}(V_{\ell n} - V_{\ell ok} + p_k) \tag{10}$$

$$rV_{\ell n} = \gamma_u(V_{hn} - V_{\ell n}), \tag{11}$$

for all $k \in \{1, \ldots, K\}$.

3.1.3 Steady-state Distribution of Types

We now provide the equations characterizing the steady-state distribution of investors' types. First, of course, all assets are being held and the mass of investors is equal to one:

$$
\begin{aligned}
s_k &= \mu_{\ell o k} + \mu_{hok} && (12)\\
1 &= \sum_{k=1}^{K} (\mu_{\ell o k} + \mu_{hok}) + \mu_{hn} + \mu_{\ell n}. && (13)
\end{aligned}
$$

Second, in a steady state, the inflow and outflow of investors in each type is zero. For example, for hn investors, we have

$$
\gamma_u \mu_{\ell n} = \gamma_d \mu_{hn} + \sum_{k=1}^{K} 2\lambda \nu_k \mu_{hn} \mu_{\ell o k}. \tag{14}
$$

The left-hand side is the flow of ℓn investors who switch from a low valuation to a high valuation, transiting to the hn type. The first term on the right-hand side, $\gamma_d \mu_{hn}$, is the flow of hn investors who switch to a low valuation. The second term is the flow of hn investors who meet sellers of some asset $k \in \{1, \ldots, K\}$ and buy an asset. Similarly, for $\ell o k$ investors,

$$
\gamma_d \mu_{hok} \quad = \quad \gamma_u \mu_{\ell o k} + 2\lambda \nu_k \mu_{hn} \mu_{\ell o k}, \tag{15}
$$

for $k \in \{1, \ldots, K\}$. Lastly, similar calculations (see Appendix A.1) show that the inflow-outflow equations for investors of types ℓn and hok are the same as (14) and (15).

3.1.4 Definition

We can now define:

Definition 2. *A steady-state symmetric equilibrium is a collection $p = (p_1, \ldots, p_K)$ of prices, a collection $V = (V_{hn}, V_{hok}, V_{\ell o k}, V_{\ell n})_{1 \le k \le K}$ of continuation utilities, a distribution $\mu = (\mu_{hn}, \mu_{hok}, \mu_{\ell o k}, \mu_{\ell n})_{1 \le k \le K}$ of types, and a trading specialists allocation, $\nu \gg 0$, such that*

(i) Steady-State: Given ν, μ solves the system (12)-(15).

(ii) Optimality: Given ν, μ, and p, V and $(\tilde{\nu}_1, \ldots, \tilde{\nu}_K) = \nu$ solve the system (8)-(11) of Bellman equations. The no-swap condition (7) holds for all $(k, j) \in \{1, \ldots, K\}^2$.

(iii) Pricing: the prices satisfy equation (5), for all $k \in \{1, \ldots, K\}$.

Although, in a symmetric equilibrium, buyers search for all assets simultaneously, they do not search all assets with identical intensity: in general, they will find it optimal to allocate different measures $\nu_k \neq \nu_j$ of trading specialists to the simultaneous search of two different assets $k \neq j$.

Note that a symmetric equilibrium has two specific properties: there are no swap and all assets are searched, that is $\nu \gg 0$.[10] In particular, since (8) is a linear program, $\nu \gg 0$ implies that hn investors are indifferent between searching for any two assets. Hence, the first-order condition of the hn investor's problem, (8), is

$$2\lambda\mu_{\ell ok}(V_{hok} - V_{hn} - p_k) = 2\lambda\mu_{\ell oj}(V_{hoj} - V_{hn} - p_j) \tag{16}$$

$$\iff \quad 2\lambda\mu_{\ell ok}(1-q)(\Delta V_{hk} - \Delta V_{\ell k}) = 2\lambda\mu_{\ell oj}(1-q)(\Delta V_{hj} - \Delta V_{\ell j}), \tag{17}$$

for all $(k,j) \in \{1,\ldots,K\}^2$, and where (17) follows from substituting (5) into (16). This first-order condition reflects "search indifference," meaning that the marginal value of allocating an additional trading specialist on a given asset is equated across assets. This marginal value is decomposed as follows: a trading specialists finds a seller of asset k with Poisson intensity $2\lambda\mu_{\ell ok}$. Then, the buyer receives a fraction $1-q$ of the transaction surplus $\Delta V_{hk} - \Delta V_{\ell k}$.

The total transaction surplus may be interpreted as a bid-ask spread, in the following sense. Suppose that the seller's bargaining power is a random variable with support $[0,1]$ and mean q, independently distributed across encounters. Then, the maximum buying price (the ask) is ΔV_{hk} and the minimum selling price price (the bid) is $\Delta V_{\ell k}$. The average price of asset k is $p_k = q\Delta V_{hk} + (1-q)\Delta V_{\ell k}$. Following this interpretation, condition (17) means that an asset that is easier to find (with a larger $\mu_{\ell ok}$) has a narrower bid-ask spread. This also suggests a negative relationship between liquidity and bid-ask spread.

3.1.5 Existence

This section provides technical conditions under which a symmetric equilibrium exists and is unique. It first analyzes the steady-state distribution of types. Second, in order to prove the existence of an equilibrium, it studies the indifference conditions (17).

First, the study of (12)-(15) presented in Appendix A.1 shows the following proposition.

[10] In Section 4.2, we show that there can be asymmetric equilibria. For instance, with two assets $k \in \{1,2\}$, for some parameter values one can construct equilibria in which buyers allocate all of their trading specialists to the search of, say, asset 1. In order to sell the other asset, 2, an investor conducts two consecutive transactions: she first swaps asset 2 for asset 1, and then she sells asset 1 to some buyer.

Proposition 1. *Given an allocation ν of trading specialist, the system (12)-(15) has a unique solution $\mu = (\mu_{hn}, \mu_{hok}, \mu_{\ell n}, \mu_{\ell ok})_{1\le k\le K} \in [0,1]^{2K+2}$.*

The Bellman equations can be simplified as follows. First, one defines the marginal value of allocating a trading specialist to the the search of asset k,

$$W_k \equiv 2\lambda\mu_{\ell ok}(1-q)(\Delta V_{hk} - \Delta V_{\ell k}), \tag{18}$$

for all $k \in \{1, \dots, K\}$. Clearly, the "search indifference" marginal conditions (17) can be written as

$$W_k = W, \tag{19}$$

for all $k \in \{1, \dots, K\}$, and for some positive constant W to be determined. Substituting (19) into equation (8), combining the Bellman equations (8) through (10), and using the pricing equation (5) one finds that

$$rW_k = 2\lambda(1-q)\mu_{\ell ok}x - (\gamma_u + \gamma_d + 2\lambda\nu_k q\mu_{hn})\,W_k - 2\lambda(1-q)\mu_{\ell ok}W, \tag{20}$$

for all $k \in \{1, \dots, K\}$. Replacing $\mu_{hok} = s_k - \mu_{\ell ok}$ in equation (15), we find that

$$2\lambda\nu_k\mu_{hn} = \frac{\gamma_d s_k}{\mu_{\ell ok}} - (\gamma_d + \gamma_u). \tag{21}$$

Substituting (21) into (20), using (19) and rearranging gives

$$\frac{2\lambda\gamma_d s_k q}{(1-q)x}\frac{1}{(2\lambda\mu_{\ell ok})^2} + \frac{r + (1-q)(\gamma_d+\gamma_u)}{(1-q)x}\frac{1}{2\lambda\mu_{\ell ok}} + \frac{1}{x} = \frac{1}{W}. \tag{22}$$

This quadratic equation allows one to write $2\lambda\mu_{\ell ok} = m_k(W)$, for some $W < x$, and for some continuous and increasing function $m_k(\,\cdot\,)$.

Now, the steady-state measure of high-valuation investors is equal to the stationary probability of being in a state of high valuation[11]

$$\mu_{hn} + \sum_{k=1}^{K}\mu_{hok} = \frac{\gamma_u}{\gamma_u+\gamma_d}. \tag{23}$$

Combining (23) with (12) shows

$$\mu_{hn} = \frac{\gamma_u}{\gamma_u+\gamma_d} - S + \sum_{k=1}^{K}\mu_{\ell ok}. \tag{24}$$

[11]This can also be shown by summing equation (15) over $k \in \{1, \dots, K\}$, subtracting equation (14) and using (13).

Substituting (24) into (21) gives

$$\nu_k \left[2\lambda \left(\frac{\gamma_u}{\gamma_u + \gamma_d} - S \right) + \sum_{k=1}^{K} m_k(W) \right] = \frac{2\lambda \gamma_d s_k}{m_k(W)} - (\gamma_d + \gamma_u), \tag{25}$$

which shows that $\sum_{k=1}^{K} \nu_k = 1$ only if

$$2\lambda \left(\frac{\gamma_u}{\gamma_u + \gamma_d} - S \right) + \sum_{k=1}^{K} m_k(W) - \sum_{k=1}^{K} \left(\frac{2\lambda \gamma_d s_k}{m_k(W)} - (\gamma_d + \gamma_u) \right) = 0. \tag{26}$$

The left-hand side of (26) is strictly increasing in W because $m_k(\,\cdot\,)$ is strictly increasing for each k. Hence, (26) uniquely characterizes a candidate equilibrium W. Once W is found, the other equilibrium objects are uniquely characterized: the trading specialist allocation ν by (25), the distribution μ of types by (12)-(15), the continuation utilities V by (8)-(10), and the prices p by (5). This implies

Proposition 2 (Uniqueness)**.** *There is at most one symmetric equilibrium.*

We now show existence in two cases: when assets are almost identical, and when the asset market is almost frictionless. In order to show existence when assets are almost identical, we start by the case of identical asset characteristics, for the distribution $\hat{s} = (S/K, \dots, S/K)$ of tradeable shares. One shows the existence of a symmetric equilibrium with $\hat{\nu}_k = 1/K$, following Duffie, Gârleanu, and Pedersen [2005]. Then, one applies the Implicit Function Theorem to equation (26), showing existence in a neighborhood of this equilibrium.

Proposition 3 (Existence with almost-identical Assets)**.** *Let $\hat{s} = (S/K, \dots, S/K)$. Then, there is a neighborhood $N \subset \mathbb{R}_+^K$ of $\hat{s}$, such that, for all $s \in N$, there is a symmetric equilibrium.*

The proof shows in particular that, if asset characteristics are sufficiently homogenous, ℓok investors are not searching for swaps. This follows from the fact that the net utility of swapping two assets with nearly identical characteristics is close to zero, and turns out to be strictly less than the net utility of searching for an outright sale.

We proceed to show existence when investors' search intensity λ goes to infinity and the search market becomes almost frictionless, following the solution method of Vayanos and Weill [2007]. The almost-frictionless economy may provide a useful description of a financial market, where search frictions are often believed to be small. Another reason for studying the almost-frictionless economy is that an equilibrium can be calculated in closed form, and that the solution technique extends to the case of a general matching function (see Section 4.1).

Proposition 4 (Existence in an almost-frictionless Market.)**.** *Suppose that the distribution* $(s_1, \ldots, s_K)$ *of tradeable shares satisfies the no-swap condition*

$$\sqrt{\max\{s_j\}} - \sqrt{\min\{s_k\}} < \sqrt{\max\{s_j\}} \frac{B}{B + \sqrt{\max\{s_j\}}}, \tag{27}$$

where

$$B \equiv \left(\frac{\gamma_u}{\gamma_u + \gamma_d} - S \right) \left(\sum_{i=1}^{K} \sqrt{s_i} \right)^{-1}.$$

Then there exists some $\bar{\lambda} \in \mathbb{R}_+$ *such that a symmetric equilibrium exists for all* $\lambda > \bar{\lambda}$*.*

The Proposition shows in particular that, asymptotically when $\lambda \to \infty$, the no-swap condition (7) is equivalent to the simple condition (27) on asset supplies. In order to interpret this condition, recall that a seller of an asset, say asset j, trades off between searching for an outright sale, or searching for a swap. As will be shown formally in the next section, in equilibrium assets in smaller supply take longer to sell. In particular, if the supply s_j of an asset is small enough relative to that of other assets, then a seller will find it optimal not to search for an outright sale with a buyer. Instead, a seller will prefer to swap her small-supply asset for some large-supply asset, and then re-sell the large supply asset to some buyer. This implies in turns that, in order for the no-swap condition to hold, asset supplies must be sufficiently similar.

3.2 Liquidity-returns Relationships

In this section we show that, in equilibrium, cross-sectional variation in tradeable shares creates a negative relationship between asset returns and liquidity.[12] Namely, consistent with qualitative evidence, equilibrium asset returns are negatively related with trading volume or turnover, and positively related with bid-ask spread.

[12]The same solution method can be applied to assets which are heterogenous in other dimensions. For example, one could consider cross-sectional variations in dividends, holding costs, and make asset heterogenous in terms of search costs. These extensions of our model are available upon request.

In contrast with the Walrasian models of Amihud and Mendelson [1986], Constantinides [1986], Vayanos [1998], and Huang [2003], our equilibrium cross-sectional variation in asset returns is not explained by an exogenously specified cross-sectional variation in transaction costs. Instead, in this model, because of search frictions, investors cannot find buyers and sellers of specific assets instantly, and because investors are impatient, the likelihoods of finding those buyers and sellers in a short time are reflected in prices. One may view cross-sectional variation in the likelihood of finding buyers and sellers as the natural counterpart of cross-sectional variation in transaction costs. This cross-sectional variation is not, however, exogenously specified. Rather, it arises endogenously and is explained by the distribution $s = (s_1, s_2, \ldots, s_K)$ of tradeable shares.

3.2.1 Three Equations

The analysis is based on three equations. The first one is the pricing equation (5), which can be written

$$p_k = \Delta V_{hk} - (1-q)(\Delta V_{hk} - \Delta V_{\ell k}). \tag{28}$$

Subtracting the Bellman equations (8) from (9) gives an expression for the reservation value ΔV_{hk} which, when substituted in (28), gives

$$rp_k \;\; = \;\; \delta - W - \gamma_d(\Delta V_{hk} - \Delta V_{\ell k}) - r(1-q)(\Delta V_{hk} - \Delta V_{\ell k}) \tag{29}$$

This equation breaks the price p_k of asset k into four components. The first, δ, is the flow value of dividend payments. The second component, W, is the flow value of searching for an asset. An hn investor obtains this discount because he has the option of not buying asset k and continuing his search. The third component, $\gamma_d(\Delta V_{hk} - \Delta V_{\ell k})$, is the instantaneous cost of switching to the low valuation, and not being able to sell the asset instantly. The last component is the bargaining discount.

Plugging the search indifference condition $W = 2\lambda\mu_{\ell ok}\,(\Delta V_{\ell k} - \Delta V_{hk})$ into equation (29), we find

$$p_k = \frac{\delta}{r} - \frac{W}{r} - \left(1 + \frac{\gamma_d}{r(1-q)}\right)\frac{W}{(2\lambda\mu_{\ell ok})}. \tag{30}$$

The right-hand side is increasing in $\mu_{\ell ok}$. In other words, an asset that is easier to find (has larger $\mu_{\ell ok}$) is sold at a higher price.

The second equation is the indifference condition (22), which is of the form

$$A s_k \frac{1}{\mu_{\ell ok}^2} + B \frac{1}{\mu_{\ell ok}} + C = \frac{1}{W}, \tag{31}$$

for some positive constants A, B, and C, which do not depend on k. This equation relates the measure $\mu_{\ell ok}$ of sellers to the measure s_k of tradeable shares.

The third equation is easily derived from (21), and relates the allocation ν_k of trading specialists to the measure $\mu_{\ell ok}$ of sellers and the measure s_k of tradeable shares:

$$\frac{\mu_{\ell ok}}{s_k} = \frac{\gamma_d}{\gamma_d + \gamma_u + 2\lambda \nu_k \mu_{hn}}. \tag{32}$$

The quantity $2\lambda\nu_k\mu_{hn}$ has several interpretations. First, it represents the demand side of the market. The larger is ν_k, the more search occurs for asset k, and the easier it is to sell this asset. It is natural to ask whether an asset that is easier to sell is also easier to find. That is, can one view $2\lambda\nu_k\mu_{hn}$ as an increasing function of $\mu_{\ell ok}$? Equation (32) shows that the answer depends on the number s_k of tradeable shares, and is thus indeterminate at this stage of the analysis.

Second, $2\lambda\nu_k\mu_{hn}$ is negatively related to the mean holding period of asset k. The holding period of a *hok* investor is some stopping time τ_h, decomposed as follows. The investor holds the asset k until she switches to a state of low valuation at some time $t + \tau_d$, where τ_d is an exponentially distributed stopping time with parameter γ_d. Then, she either meets a buyer or switches back to a high valuation at some time $t + \tau_d + \min\{\tau_b, \tau_u\}$, where τ_b and τ_u are exponentially distributed stopping times with respective parameters $2\lambda\nu_k\mu_{hn}$ and γ_u. If she switches back to a high valuation utility, then her mean holding period is some stopping time $\tilde{\tau}_h$. Hence,

$$\tau_h = \tau_d + \mathbb{I}_{\{\tau_u < \tau_b\}}(\tau_u + \tilde{\tau}_h) + \mathbb{I}_{\{\tau_b \leq \tau_u\}}\tau_b = \tau_d + \min\{\tau_u, \tau_b\} + \mathbb{I}_{\{\tau_u < \tau_b\}}\tilde{\tau}_h. \tag{33}$$

In a steady-state equilibrium, $\tilde{\tau}_h$ and τ_h are identically distributed. Furthermore, all the above stopping times are pairwise independent. Taking expectations of both sides of (33), and using the fact that τ_h and $\tilde{\tau}_h$ are identically distributed, one finds that

$$E(\tau_h) = \frac{1}{\gamma_d} + \frac{1}{\gamma_u + 2\lambda\nu_k\mu_{hn}} + \frac{\gamma_u}{\gamma_u + 2\lambda\nu_k\mu_{hn}} E(\tau_h)$$

and therefore

$$E(\tau_h) = \frac{1}{\gamma_d} + \frac{1}{2\lambda\nu_k\mu_{hn}}\left(1 + \frac{\gamma_u}{\gamma_d}\right). \tag{34}$$

This shows that the mean holding period $E(\tau_h)$ is a decreasing function of $2\lambda\nu_k\mu_{\ell n}$.

3.2.2 Returns and Liquidity Proxies

Equation (31) has the form

$$F(s_k, \mu_{\ell ok}) = \frac{1}{W}, \tag{35}$$

for some function $F(\,\cdot\,,\,\cdot\,)$ that is increasing in s_k and decreasing in $\mu_{\ell ok}$. This implies that $\mu_{\ell ok}$ is increasing in s_k. In other words, an asset with more tradeable shares is easier to find, is sold at a higher price, and has a lower return $R_k = \delta/p_k$. In order to derive a relationship between the number s_k of tradeable shares and the mean holding period (34), one writes equation (31) as

$$G\left(s_k, \frac{\mu_{\ell ok}}{s_k}\right) = \frac{1}{W}, \tag{36}$$

for some function $G(\,\cdot\,,\,\cdot\,)$ that is decreasing in s_k and decreasing in $\mu_{\ell ok}/s_k$. This implies that $\mu_{\ell ok}/s_k$ is a decreasing function of s_k. From (32), it follows that $2\lambda\nu_k\mu_{hn}$ is an increasing function of s_k. In other words, an asset with more tradeable shares has a shorter mean holding period. Lastly, since the total rate of contact between buyers and sellers of asset k is $2\lambda\nu_k\mu_{hn}\mu_{\ell ok}$, an asset with more tradeable shares also has a larger trading volume. The above discussion is summarized in

Proposition 5. *In equilibrium, $s_k > s_j$ implies that $\mu_{\ell ok} > \mu_{\ell oj}$, $\nu_k > \nu_j$, $p_k > p_j$, $R_k < R_j$, and $\Delta V_{hk} - \Delta V_{\ell k} < \Delta V_{hj} - \Delta V_{\ell j}$.*

In words, an asset with more tradeable shares is easier to find, easier to sell, has a higher price, a lower return, and a narrower bid-ask spread. This implies in turn that it also has a larger trading volume, a larger turnover, and a shorter mean holding period.

In contrast with Proposition 5, the one-asset model of Duffie, Gârleanu, and Pedersen [2005] implies that an asset with a larger number of tradeable shares has a *lower* price. Indeed, an increase in the number of tradeable shares results in a positive shift of the supply curve, and hence lowers the price of the asset. Similarly, in our model, a larger s_k represents a larger supply. However, a larger s_k also endogenously results in a larger demand, represented by a larger search intensity $\lambda \nu_k$. Proposition 5 shows that the "demand shift" dominates, meaning that an asset with a larger number s_k of tradeable shares has a higher price.

This model generates a positive relationship between returns and holding periods with *ex-ante* identical investors, because returns and holding periods are both negatively related to a common exogenous "liquidity" factor, the number of tradeable shares. By contrast Amihud and Mendelson [1986] take the holding period itself to be an exogenous parameter. A positive relationship between returns and holding periods also arises endogenously in general equilibrium models with transaction costs, such as those of Vayanos and Vila [1999] or Huang [2003], but for a different reason. In these models, assets can be bought and sold instantly, and an investor chooses to hold assets with larger transaction costs for a longer period. These assets, in equilibrium, have higher expected returns. In the present model, an asset cannot be bought and sold instantly, and an asset with a higher return takes longer to sell, and thus has a longer mean holding period.

3.3 Empirical Evidence

In the model, investors are constrained to use a different group of trading specialists for each asset. As discussed before, in practice, stock trading and analysis is typically specialized by industry or by asset class. Thus, the empirical counterpart of an asset in our model may be, say, an asset class or a portfolios of stocks from a particular industry. This means that the model provides predictions regarding the relationship between liquidity and expected returns across, say, industry portfolios as opposed to individual stocks. By contrast, information-based market microstructure models would typically provide predictions on the liquidity-return relationship across individual stocks.[13] This observation also suggests in turn that existing empirical studies of the negative relationship between cross-sectional returns and liquidity proxies may not provide a direct test of the model's prediction. Indeed, most empirical studies focus on individual stocks (e.g. Brennan, Chordia, and Subrahmanyam [1998] or Haugen and Baker [1996]), as opposed to portfolios sorted by industries. These studies may still provide an indirect test of our model, as the liquidity level of an individual

[13] In these models, insiders are more likely to have private information about the cash flow of a particular stock.

stock is, in part, explained by the liquidity level of its industry, as documented by Chordia, Roll, and Subrahmanyam [2000], Table 9.[14]

Some empirical studies suggest that assets with a larger number of tradeable shares are more liquid. For instance, Chan, Chan, and Fong [2004] study the impact of a reduction in the number of tradeable shares on asset liquidity. In August 1998, the Hong Kong monetary authority opposed a speculative attack by aggressively buying the 33 stocks of the Hang Seng 33 Index (HSI 33). The monetary authority absorbed about 7.3% of HSI 33 market capitalization and held these stocks for a long time period, resulting in a reduction in the number of tradeable shares of these stocks. The authors show that, relative to some control group with no reduction, the HSI stocks experienced a decrease in liquidity. Zhang, Tian, and Wirjanto [2007] also document a negative relationship between tradeable shares and risk-adjusted cross-sectional returns. They study monthly returns on stocks traded in the Shanghai Stock Exchange (SHSE) and the Senzhen Stock Exchange (SZSE), from July 1993 to December 2001. They measure the tradeable shares by the market values of shares which can be traded by domestic and foreign investors. During the sample period, non-tradeable shares represent over 70% of market capitalization, and are owned by the state or are restricted institutional shares.

4 Extensions

This section presents two extensions of the model. First, we show that our existence result extends to the case of a general matching function. Second, we study an asymmetric equilibrium where buyers choose to search for one of two assets.

4.1 General Matching Function

We consider the same setup as before, but with a general matching function. Namely, we let the instantaneous rate of contact between buyers and sellers of asset k be

$$M\left(2\lambda\mu_{\ell ok}, \nu_k \mu_{hn}\right) \equiv M_k, \tag{37}$$

for some matching function $M(\,\cdot\,,\,\cdot\,)$. Recall that $\nu_k \mu_{hn}$ represents the aggregate number of trading specialists searching for asset k. The search intensity parameter, λ, multiplies

[14] In addition, these authors also show that, in the time series, the liquidity of an individual stock co-varies with that of its industry. See also Huberman and Halka [2001].

the first argument of the matching function (37) because it helps derive asymptotic results when search frictions vanish. Note that the model of the previous section obtains when $M(a, b) = ab$.

We assume that the matching function $M(\cdot, \cdot)$ is "well behaved:" it is twice continuously differentiable, increasing in both arguments, and such that, for all $(a, b) \in \mathbb{R}_+$, $M(0, b) = M(a, 0) = 0$. Importantly, in order for the no-swap condition to hold, we assume that $\partial M/\partial a(0, b) < \infty$. Lastly, we also assume that search frictions vanish when $\lambda \to \infty$: for all $b > 0$, $M(a, b) \to \infty$ as $a \to \infty$.

Given the matching function of equation (37), the Poisson intensity with which a single buyer finds sellers is given by $M_k/(\nu_k \mu_{hn})$, the aggregate rate of contact divided by the measure of trading specialists searching for asset k. Similarly the intensity with which a seller finds buyers is $M_k/\mu_{\ell ok}$. An equilibrium is defined as before and the equilibrium equations are derived in Appendix A.3.1. We start with an existence result when $\lambda \to \infty$.

Proposition 6 (General Matching Function)**.** *Let $\hat{s} = (S/K, \ldots, S/K)$. Then, there exists some neighborhood $N \subseteq R_+^K$ of $\hat{s}$ and some $\bar{\lambda} > 0$ such that, for all $s \in N$ and $\lambda > \bar{\lambda}$, there exists a unique symmetric equilibrium.*

Next, we provide an asymptotic approximation of equilibrium objects.

Proposition 7 (Asymptotic Expansions)**.** *There exists $(\bar{m}_{\ell o1}, \ldots, \bar{m}_{\ell oK}) \in \mathbb{R}_+^K$ such that*

$$\mu_{\ell ok} = \frac{\bar{m}_{\ell ok}}{2\lambda} + o(1/(2\lambda)) \tag{38}$$

$$\nu_k = \frac{\bar{m}_{\ell ok}}{\sum_{j=1}^K \bar{m}_{\ell oj}} + o(1). \tag{39}$$

In addition, $s_k > s_j$ implies $\bar{m}_{\ell ok} > \bar{m}_{\ell oj}$.

This Proposition shows that, asymptotically, the measures of sellers goes to zero, meaning that assets are almost perfectly allocated to high-valuation investors.[15] In addition, it shows that the ratio $2\lambda\mu_{\ell ok}/\nu_k = \bar{m}_{\ell ok}/\nu_k$ is asymptotically constant in the cross-section: this

[15]Note that the measure $\mu_{\ell ok}$ of sellers goes to zero while the measure μ_{hn} of buyers stays bounded away from zero. Hence, as it is standard in such frictionless limit, sellers are able to contact buyers almost instantly but buyers take finite times, on average, to contact sellers. This asymmetry in contact times activates Bertrand competition among buyers but not among sellers, and drives the asset price to its Walrasian value of δ/r.

illustrates the sense in which the distribution of asset demands, represented by the ν_k's, adjusts to compensate for differences in asset supplies, represented by the $\bar{m}_{\ell ok}$'s.

Whether a larger asset supply and demand, $\bar{m}_{\ell ok}$ and ν_k, result in an increase in liquidity depends on the shape of the matching function. According to the Proposition, the ratio $\bar{m}_{\ell ok}/\nu_k$ is asymptotically constant in the cross section. Hence, the intensity $M_k/\mu_{\ell ok}$ with which a seller of asset k meets a buyer, and the intensity $M_k/(\nu_k \mu_{hn})$ with which a buyer meets a seller, are both asymptotically proportional to

$$\frac{M(\bar{m}_{\ell ok}, \omega \bar{m}_{\ell ok})}{\bar{m}_{\ell ok}},$$

for some constant of proportionality ω. Hence the intensity of contact of buyers and sellers will increase in $\bar{m}_{\ell ok}$ and s_k if the matching function has increasing returns to scale, and will decrease in $\bar{m}_{\ell ok}$ and s_k otherwise.[16] Intuitively, when the matching function has decreasing returns to scale, an increase in trading activity (higher M_k) creates a congestion externality, and ends up decreasing the contact intensities $M_k/\mu_{\ell ok}$ and $M_k/(\nu_k \mu_{hn})$. These effects imply natural cross-sectional relationships between returns and liquidity proxies:

Proposition 8 (Liquidity-return Relationships). *Consider the equilibrium with small search frictions characterized in Proposition 6. Then $s_k > s_j$ implies that $\mu_{\ell ok} > \mu_{\ell oj}$, $\nu_k > \nu_j$, and*

(i) if the matching function has increasing returns to scale, that asset k is more liquid than asset j. That is, $p_k > p_j$, $R_k < R_j$, $\Delta V_{hk} - \Delta V_{\ell k} < \Delta V_{hj} - \Delta V_{\ell j}$, and the dollar turnover of asset k is larger than that of asset j.

(ii) if the matching function has decreasing returns to scale, that asset k is less liquid than asset j. That is, $p_k < p_j$, $R_k > R_j$, $\Delta V_{hk} - \Delta V_{\ell k} > \Delta V_{hj} - \Delta V_{\ell j}$, and the dollar turnover of asset k is smaller than that of asset j.

The Proposition shows that some qualitative properties of an equilibrium are the same under increasing or decreasing returns: in both cases, cross-sectional returns are negatively related to turnover and positively related to bid-ask spread. Other properties, however, are different: liquidity is positively related to the number of tradeable shares if the matching function has increasing returns, and negatively related to the number of tradeable shares

[16] Following the definition of Pissarides [2000], the matching function is said to have increasing returns to scale if, for all $(a, b) \in \mathbb{R}_+$ and $\psi > 1$, $M(\psi a, \psi b) > \psi M(a, b)$, decreasing returns to scale if the reverse inequality holds, and constant returns if it holds with equality.

otherwise. Hence, according to the model, the sign of the cross-sectional relationship between liquidity and tradeable shares allows to empirically distinguish between increasing and decreasing returns in matching. The Proposition suggests to proxy liquidity by bid-ask spread, dollar turnover, or risk-adjusted return. Measuring the number of tradeable shares is, admittedly, a delicate issue because one has to decide which shares are available for investment, and which ones are not. The next Proposition shows that, according to the present model, the number of tradeable shares can be proxied by the dollar trading volume:

Proposition 9. *Consider the equilibrium with small search frictions characterized in Proposition 6. Then, the dollar trading volume of asset $k \in \{1, \ldots, K\}$ is*

$$(\gamma_d \delta / r)\, s_k + o(1),$$

and is therefore asymptotically proportional to the number s_k of tradeable shares.

4.2 An Asymmetric Equilibrium

So far we restricted attention to equilibria where buyers allocate a positive measure $\nu_k > 0$ of trading specialists to the search of all assets. In this section we show that there are also other kinds of equilibria: we consider a two assets economy, $K = 2$, and we construct an equilibrium where buyers search only for asset 1, i.e. $\nu_1 = 1$ and $\nu_2 = 0$. This does not mean, however, that a seller of asset 2 has to hold her asset forever. Instead, a seller of asset 2 can sell her asset by conducting two consecutive trades: she first searches for an $ho1$ investor in order to swap asset 2 for asset 1. Then she searches for an hn investor in order to sell asset 1.

Proposition 10 (Existence of an Asymmetric Equilibrium)**.** *Suppose that $K = 2$. Then there is some $\bar{\lambda}$ such that, for all $\lambda > \bar{\lambda}$, there exists an asymmetric equilibrium in which buyers only search for asset $k = 1$.*

We proceed with some pricing implications. We define the price of asset $k \in \{1, 2\}$ as before: $p_k = q\Delta V_{hk} + (1-q)\Delta V_{\ell k}$. When an hn meets an $\ell o1$ investor, she purchases asset 1 at price p_1. When a $\ell o2$ investor meets a $ho1$ investor, they swap assets for a fee. Assuming that the bargaining power of the $\ell o2$ investor is $q \in [0, 1]$, then one shows easily that the $\ell o2$ investor pays a fee of $p_1 - p_2$ to the $ho1$ investor.

Proposition 11 (Pricing in an Asymmetric Equilibrium)**.** *Consider the asymmetric equilibrium with small frictions of Proposition 10. Then,*

(i) $p_1 > p_2$.

(ii) There exist parameter values such that $p_1 > \delta/r$*, i.e. the price of asset 1 is greater than the Walrasian price.*

(iii) Holding $S = s_1 + s_2$ *of tradeable shares constant, the price of asset 1 decreases, and the price of asset 2 increases, in* s_1*.*

The first point of the Proposition shows that, in an asymmetric equilibrium, asset 1 is more expensive than asset 2 because it can be sold more quickly. The second point follows because high-valuation holders of asset 1, $ho1$, earn a "convenience yield" on a search market: they can provide liquidity services to sellers of asset 2, who seek to swap their asset in order to unwind their positions. The present value of the associated swap fees are capitalized into the price p_1 of asset 1 and, in some case, can raise p_1 above the Walrasian price. This would occur, for instance, if the supply s_1 of asset 1 is small enough: in that case, sellers of asset 2 have a hard time finding asset 1 for a swap, which make them willing to pay a high fee. The negative effect of supply on swap fees explains the last point of the Proposition. Indeed, increasing s_1 makes it more easy to find asset s_1 for a swap, reduces swap fees, decreases the price of asset 1 and increases that of asset 2. Note that these effects of supply on convenience yields are similar to the ones analyzed by Duffie, Gârleanu, and Pedersen [2002] and Vayanos and Weill [2007] in the market for borrowing and lending securities.

5 Conclusion

This paper uses a search-theoretic model to study the impact of heterogeneity in tradeable shares on the cross section of asset returns. Although the search technology is the same for all assets, heterogeneous trading costs arise endogenously. In equilibrium, an asset return is negatively related to its number of tradeable shares, its turnover, its trading volume, and it is positively related to its bid ask spread.

A Proofs

A.1 Proof of Proposition 1

For a given allocation ν of trading specialists, the distribution of types solves the system of inflow-outflow equations:

$$\begin{aligned}
\gamma_u \mu_{\ell n} &= \gamma_d \mu_{hn} + \sum_{k=1}^{K} 2\lambda \nu_k \mu_{hn} \mu_{\ell ok} && (40)\\
\gamma_u \mu_{\ell ok} + 2\lambda \nu_k \mu_{hn} \mu_{\ell ok} &= \gamma_d \mu_{hok} && (41)\\
\gamma_d \mu_{hn} + 2\sum_{k=1}^{K} 2\lambda \nu_k \mu_{hn} \mu_{\ell ok} &= \gamma_u \mu_{\ell n} && (42)\\
\gamma_d \mu_{hok} &= \gamma_u \mu_{\ell ok} + 2\lambda \nu_k \mu_{hn} \mu_{\ell ok} && (43)
\end{aligned}$$

Equation (40) is for investors of type hn, equation (41) for hok , equation (42) for ℓn, and (43) for ℓok. The distribution of types must also satisfy

$$\begin{aligned}
s_k &= \mu_{\ell ok} + \mu_{hok} && (44)\\
1 &= \sum_{k=1}^{K} (\mu_{\ell ok} + \mu_{hok}) + \mu_{hn} + \mu_{\ell n}, && (45)
\end{aligned}$$

Since equation (44) implies that the sum of (41) and (43) is zero, one can eliminate (41). Similarly, since equation (45) implies that the sum of equations (40) to (43) is zero, one can eliminate (42), and obtains the equivalent system

$$\begin{aligned}
\gamma_d s_k &= \gamma \mu_{\ell ok} + 2\lambda \nu_k \mu_{hn} \mu_{\ell ok} && (46)\\
\gamma_u (1-S) &= \gamma \mu_{hn} + 2\sum_{k=1}^{K} \lambda \nu_k \mu_{hn} \mu_{\ell ok} && (47)\\
s_k &= \mu_{\ell ok} + \mu_{hok} && (48)\\
1 - S &= \mu_{hn} + \mu_{\ell n}, && (49)
\end{aligned}$$

for $k \in \{1, \ldots, K\}$, and where $\gamma \equiv \gamma_u + \gamma_d$. Adding equations (46) over $k \in \{1, \ldots, K\}$, subtracting equation (47) we find

$$\mu_{hn} = \mu_{\ell o} + y - S, \tag{50}$$

where $\mu_{\ell o} \equiv \sum_{k=1}^{K} \mu_{\ell ok}$, and $y \equiv \gamma_u/(\gamma_d + \gamma_u)$. Replacing this last equation in (46) gives

$$\mu_{\ell ok} = \frac{\gamma_d s_k}{\gamma_d + \gamma_u + 2\lambda \nu_k \left(\mu_{\ell o} + y - S\right)}. \tag{51}$$

Summing equations (51) over k, one obtains the one equation in one unknown problem

$$\mu_{\ell o} - \sum_{k=1}^{K} \frac{\gamma_d s_k}{\gamma_d + \gamma_u + 2\lambda \nu_k \left(\mu_{\ell o} + y - S\right)} = 0. \tag{52}$$

The left-hand side of this equation is increasing in $\mu_{\ell o}$, is negative at $\mu_{\ell o} = 0$, and is positive for $\mu_{\ell o}$ large enough; thus, it has a unique solution. Once the solution $\mu_{\ell o}$ is found, $\mu_{\ell ok}$ is uniquely determined by (51), μ_{hn} by (50), and finally μ_{hok} and $\mu_{\ell n}$ by (48) and (49). This characterizes a unique candidate steady state. Since the steady-state fractions sum to one by construction, one only needs to show that they are positive as follows: The left-hand side of (52) is positive when evaluated at $\mu_{\ell o} = S$ and $1-y$; it is negative when evaluated at $S-y$. Since the left-hand side of (52) is increasing, this shows that

$$S - y < \mu_{\ell o} < \min\{S, 1-y\}. \tag{53}$$

Next, $s - y < \mu_{\ell o}$ implies that $\mu_{hn} > 0$ and that $\mu_{\ell ok} < s_k$. Finally, $\mu_{\ell o} < 1 - y$ implies that $\mu_{hn} < 1 - S$ and that $0 < \mu_{\ell n} < 1$.

A.2 Proof of Proposition 3

If the assets all have the same number of tradeable share $s_k = \hat{s}$ for some $\hat{s} > 0$, it is natural to guess that there is a symmetric equilibrium, with $\hat{\mu}_{\ell ok} = \hat{\mu}_{\ell o}/K$ and $\hat{\nu}_k = 1/K$. The equilibrium equations are those of Duffie et al. [2005], with "λ" there being replaced here by "λ/K." Their results imply that investors' values are strictly positive, and that there are strictly positive gains from trade between investors of types hn and ℓok. Furthermore, since assets have identical characteristics, there is no gain from swapping assets. Thus, ℓok investors strictly prefer searching for an outright sale with an hn investor to searching for a swap with an hoj investor, for all $j \in \{1, \ldots, K\}$. Since the left-hand side of (26) is strictly increasing in W, the Implicit Function Theorem (see Taylor and Mann [1983], chapter 12) can be applied: This provides a neighborhood $N \subset \mathbb{R}_+^K$ of $\hat{s}$, such that, for all $s \in N$, there exists a candidate equilibrium $W = h(s)$, for some continuous function $h(\,\cdot\,)$. The other candidate equilibrium objects (V, μ, λ) are easily expressed as continuous functions of W and thus as continuous functions of s. The search-indifference marginal conditions (19) are satisfied by construction. The no swap condition (7) as well as all other relevant inequalities hold by continuity.

A.3 Proof of Propositions 4, 6, 7, and 8

In this appendix we study an equilibrium when the search market is almost frictionless and when buyers and seller meet according to a general matching function. In section A.3.1 we provide the equilibrium equations. In section A.3.2 we prove the existence result. In section A.3.3, we study asset prices.

A.3.1 Equilibrium Equations

The instantaneous rate of contact between buyers and sellers of asset k be represented by the "matching function"

$$M_k \equiv M\left(2\lambda \nu_k \mu_{\ell ok}, \mu_{hn}\right). \tag{54}$$

Buyers establish contact with sellers at Poisson arrival times with intensity $M_k/(\nu_k \mu_{hn})$ and sellers establish contact with buyers at Poisson arrival times with intensity $M_k/\mu_{\ell ok}$. We let $\varepsilon = 1/(2\lambda)$ and $m_{\ell ok} \equiv 2\lambda \mu_{\ell ok}$ so that $\mu_{\ell ok} = \varepsilon m_{\ell ok}$. We start by re-stating the equilibrium conditions in terms of these new notations. First, proceeding as with equations (46) and (47) of Appendix A.1, one shows that the measures $m_{\ell ok}$ and μ_{hn} solve

$$\gamma_d s_k = \gamma \varepsilon m_{\ell ok} + M_k, \tag{55}$$

$$\gamma_u (1 - S) = \gamma \mu_{hn} + \sum_{k=1}^{K} M_k \tag{56}$$

for $k \in \{1, \ldots, K\}$ and where $\gamma \equiv \gamma_u + \gamma_d$. We also have that $\mu_{hok} = s_k - \mu_{\ell ok}$ and $\mu_{\ell n} = 1 - S - \mu_{hn}$. Summing (55) over $k \in \{1, \ldots K\}$ and subtracting (56), one finds:

$$\mu_{hn} = y - S + \varepsilon \sum_{k=1}^{K} m_{\ell ok}, \tag{57}$$

where $y \equiv \gamma_u/(\gamma_u + \gamma_d)$. Second, the Bellman equations are

$$
\begin{aligned}
rV_{hn} &= \gamma_d (V_{\ell n} - V_{hn}) + \sum_{k=1}^{K} \tilde{\nu}_k \frac{M_k}{\nu_k \mu_{hn}} (1-q)\Sigma_k && (58)\\
rV_{hok} &= \delta + \gamma_d (V_{\ell ok} - V_{hok}) && (59)\\
rV_{\ell ok} &= \delta - x + \gamma_u (V_{hok} - V_{\ell ok}) + \frac{M_k}{\varepsilon m_{\ell ok}} q\Sigma_k && (60)\\
rV_{\ell n} &= \gamma_u (V_{hn} - V_{\ell n}), && (61)
\end{aligned}
$$

where $\Sigma_k = V_{hok} + V_{\ell n} - V_{\ell ok} - V_{hn}$ is the surplus of a transaction between a buyer and a seller. The allocation of trading specialists must satisfy the resource constraint

$$
\sum_{k=1}^{K} \nu_k = 1.
$$

Lastly, we write the indifference condition of buyers

$$
\varepsilon\omega = \frac{M_k}{\nu_k \mu_{hn}} (1-q)\Sigma_k, \tag{62}
$$

for some search indifference level $\varepsilon\omega$ to be determined in equilibrium. Now, we find an equation for Σ_k by adding equations (59) and (61), subtracting (58) and (60):

$$
r\Sigma_k = x - \gamma\Sigma_k - \sum_{j=1}^{K} \tilde{\nu}_j \frac{M_j}{\nu_j \mu_{hn}} (1-q)\Sigma_j - \frac{M_k}{\varepsilon m_{\ell ok}} q\Sigma_k. \tag{63}
$$

Now indifference condition (62) shows that $\Sigma_k = \varepsilon\omega(\nu_k \mu_{hn})/(M_k(1-q))$. Replacing this expression into (63) we obtain

$$
\begin{aligned}
& r\Sigma_k = x - \gamma\Sigma_k - \varepsilon\omega - \frac{M_k}{\varepsilon m_{\ell ok}} q\Sigma_k \\
\Rightarrow\quad & \left(r + \gamma + \frac{M_k q}{\varepsilon m_{\ell ok}}\right) \Sigma_k = x - \varepsilon\omega \\
\Rightarrow\quad & \left(r + \gamma + \frac{M_k q}{\varepsilon m_{\ell ok}}\right) \frac{\nu_k \mu_{hn}}{M_k(1-q)} \varepsilon\omega = x - \varepsilon\omega \\
\Rightarrow\quad & \left(\frac{(r+\gamma\varepsilon}{M_k} + \frac{q}{m_{\ell ok}}\right) \nu_k \mu_{hn} = (1-q)\left(\frac{x}{\omega} - \varepsilon\right) \\
\Rightarrow\quad & \nu_k \mu_{hn} = (1-q)\left(\frac{x}{\omega} - \varepsilon\right)\left(\frac{r+\gamma\varepsilon}{M_k} + \frac{q}{m_{\ell ok}}\right)^{-1} && (64)
\end{aligned}
$$

Now equation (55) shows that $M_k = \gamma_d s_k - \gamma\varepsilon m_{\ell ok}$. Plugging this back into the right-hand side of (64) we obtain

$$
\nu_k \mu_{hn} = (1-q)\left(\frac{x}{\omega} - \varepsilon\right)\phi(m_{\ell ok}), \tag{65}
$$

where

$$
\phi(m) \equiv \left[\frac{r+\gamma)\varepsilon}{\gamma_d s_k - \gamma\varepsilon m} + \frac{q}{m}\right]^{-1}.
$$

Adding up equations (65) for all $k \in \{1, \ldots, K\}$, using that $\sum_{k=1}^{K} \nu_k = 1$, we find that

$$
\mu_{hn} = (1-q)\left(\frac{x}{\omega} - \varepsilon\right)\sum_{k=1}^{K} \phi(m_{\ell ok}).
$$

Now plugging this expression into (57) we find that

$$(1-q)\left(\frac{x}{\omega}-\varepsilon\right)\sum_{k=1}^{K}\phi(m_{\ell ok}) = y - S + \varepsilon\sum_{k=1}^{K} m_{\ell ok}. \tag{66}$$

Plugging (65) into the matching function of equation (55), we find

$$\gamma_d s_k = \gamma\varepsilon m_{\ell ok} + M\left[m_{\ell ok}, (1-q)\left(\frac{x}{\omega}-\varepsilon\right)\phi(m_{\ell ok})\right]. \tag{67}$$

Equations (66) for $k \in \{1,\ldots,K\}$ and equation (67) constitute a system of $K+1$ equations in the $K+1$ unknowns ω and $(\mu_{\ell o1},\ldots,\mu_{\ell oK})$. Given a solution to this system, one construct the rest of the candidate equilibrium objects as follows. The distribution of type is given by $\mu_{hok} = s_k - \varepsilon m_{\ell ok}$, $\mu_{hn} = y - S + \varepsilon\sum_{k=1}^{K} m_{\ell ok}$, and $\mu_{\ell n} = 1 - S - \mu_{hn}$. The allocation of trading specialist is found using equation (65). One can calculate the continuation utilities using the Bellman equation, and the indifference condition holds by construction. One then need to verify that the rest of the equilibrium conditions hold.

A.3.2 Existence: Propositions 4, 6, and 7

We first show that, when $\varepsilon = 0$, equations (66) and (67) have a unique solution, $(\bar{\omega}, \bar{m}_{\ell ok})$. Second, we apply the implicit function theorem to shows that a unique solution exists for ε close to zero. Third, we verify that this solution is the basis of an equilibrium.

Step 1: the frictionless limit $\varepsilon = 0$. When $\varepsilon = 0$, $\phi(m) = m/q$. Thus, equations (66) and (67) become

$$\begin{aligned} y - S &= \frac{(1-q)x}{q}\sum_{k=1}^{K}\frac{m_{\ell ok}}{\omega} && (68)\\ \gamma_d s_k &= M\left[\omega\frac{m_{\ell ok}}{\omega}, \frac{(1-q)x}{q}\frac{m_{\ell ok}}{\omega}\right]. && (69)\end{aligned}$$

Recall that $M(a,b)$ is assumed to be strictly increasing in both its argument, differentiable, because $M(0,0) = 0$, and $M(a,b)$ goes to infinity as a and b go to infinity. It then follows that, for any $\omega > 0$, there exists a unique $m_{\ell ok}/\omega \equiv \theta_k(\omega)$ solving equation (69), for some continuous and strictly decreasing function $\theta_k(m)$. In addition $\theta_k(m)$ goes to zero as ω goes to infinity, and goes to infinity as ω goes to zero. Plugging this function back into (68), one find the one-equation-in-one-unknown problem:

$$y - S = \frac{(1-q)x}{q}\sum_{k=1}^{K}\theta_k(\omega),$$

which clearly has a unique solution $\bar{\omega} > 0$. We then define $\bar{m}_{\ell ok} \equiv \bar{\omega}\theta_k(\bar{\omega})$.

Step 2: existence in an almost frictionless search market. We start by establishing that equations (66) and (67) have a unique solution when ε is close to zero. To that end, we apply the Implicit Function Theorem to the system $H(z,\varepsilon) = 0$, where $z' \equiv (1/\omega, m_{\ell o1},\ldots,m_{\ell oK}) \in \mathbb{R}^K$ and

$$\begin{aligned} H_0(z,\varepsilon) &= (1-q)\left(\frac{x}{\omega}-\varepsilon\right)\sum_{k=1}^{K}\phi(m_{\ell ok},\varepsilon) - (y-S) - \varepsilon\sum_{k=1}^{K} m_{\ell ok}\\ H_k(z,\varepsilon) &= \gamma\varepsilon m_{\ell ok} + M\left[m_{\ell ok}, (1-q)\left(\frac{x}{\omega}-\varepsilon\right)\phi(m_{\ell ok},\varepsilon)\right] - \gamma_d s_k,\end{aligned}$$

for $k \in \{1,\ldots,K\}$, and we made it explicit that function ϕ depends on ε. We let $\bar{z}$ be the solution of $H(\bar{z},0)$ that we characterized in the previous paragraph. In order to apply the Implicit Function Theorem at $\varepsilon = 0$, we need to prove that the Jacobian $\partial_z H(\bar{z}, 0)$ is invertible. Keeping in mind that $\phi(m, 0) = m/q$ and $\phi'(m, 0) = 1/q$ when $\varepsilon = 0$, we find that, at $(z, \varepsilon) = (\bar{z}, 0)$:

$$\begin{aligned}
\frac{\partial H_0}{\partial z_0} = \frac{\partial H_0}{\partial 1/\omega} &= \frac{(1-q)x}{q} \sum_{k=1}^{K} \bar{m}_{\ell ok} \equiv a \\
\frac{\partial H_0}{\partial z_k} = \frac{\partial H_0}{\partial m_{\ell ok}} &= \frac{(1-q)x}{q} \frac{1}{\bar{\omega}} \equiv c \\
\frac{\partial H_k}{\partial z_0} = \frac{\partial H_k}{\partial 1/\omega} &= \frac{\partial M_k}{\partial \mu_{hn}} \frac{(1-q)x}{q} \bar{m}_{\ell ok} \equiv b_k \\
\frac{\partial H_k}{\partial z_k} = \frac{\partial H_k}{\partial m_{\ell ok}} &= \frac{\partial M_k}{\partial m_{\ell ok}} + \frac{\partial M_k}{\mu_{hn}} \frac{(1-q)x}{q} \frac{1}{\bar{\omega}} \equiv d_k,
\end{aligned}$$

and $\partial H_k / \partial m_{\ell oj} = 0$ for $j \neq k$. Thus, the Jacobian can be written

$$\partial_z H = \begin{bmatrix} a & ce' \\ b & \text{diag}(d) \end{bmatrix},$$

where a and c are defined above, $b' \equiv (b_1, \ldots, b_K)$, $d' \equiv (d_1, \ldots, d_K)$, and $e' = (1, \ldots, 1)$. In order to show that $\partial_z H(\bar{z}, 0)$ is invertible, suppose there is some $z' = (z_0, z_1, \ldots, z_K) \neq 0$ such that $\partial_z H(\bar{z}, 0) \cdot z = 0$. That is:

$$\begin{aligned}
az_0 + ce' z_{1K} &= 0 \\
bz_0 + \text{diag}(d) z_{1K} &= 0,
\end{aligned}$$

where $z'_{1K} \equiv (z_1, \ldots, z_K)$. The second equation implies that $z_{1K} = -\text{diag}(d)^{-1} b z_0$. Since $z \neq 0$ this implies that $z_0 \neq 0$. Plugging this back into the first equation we find that

$$az_0 = ce' \,\text{diag}(d)^{-1}\, bz_0.$$

We can simplify $z_0 \neq 0$ from both sides of the equations. Plugging the expression of a, b, c, and d, we obtain

$$\begin{aligned}
&\frac{1-q}{q} x \sum_{k=1}^{K} \bar{m}_{\ell ok} - \frac{(1-q)x}{q} \frac{1}{\bar{\omega}} \sum_{k=1}^{K} \frac{\frac{\partial M_k}{\partial \mu_{hn}} \frac{(1-q)x}{q} \bar{m}_{\ell ok}}{\frac{\partial M_k}{\partial \mu_{hn}} \frac{(1-q)x}{q} \frac{1}{\bar{\omega}} + \frac{\partial M_k}{\partial \bar{m}_{\ell ok}}} = 0 \\
\Rightarrow \quad & \sum_{k=1}^{K} \bar{m}_{\ell ok} - \sum_{k=1}^{K} \bar{m}_{\ell ok} \frac{\frac{\partial M_k}{\partial \mu_{hn}} \frac{(1-q)x}{q} \frac{1}{\bar{\omega}}}{\frac{\partial M_k}{\partial \mu_{hn}} \frac{(1-q)x}{q} \frac{1}{\bar{\omega}} + \frac{\partial M_k}{\partial \bar{m}_{\ell ok}}} = 0,
\end{aligned}$$

which is impossible because, in the second term, all the fractions multiplying $\bar{m}_{\ell ok}$ are less than one. Thus, $\partial H_z(\bar{z}, 0)$ is invertible, and an application of the Implicit Function Theorem shows that, as long as ε is close enough to zero, the system (66) and (67) of equations has a unique solution $\{m_{\ell ok}\}_{k=1}^{K}$ and ω.

Step 3: Verification. One can construct other equilibrium objects as explained above. Because the solution we have just constructed is continuous in ε, direct calculations show that the distribution of types admit the following first-order approximation when ε is close to zero:

$$\begin{aligned}
\mu_{hn} &= y - S + \varepsilon \sum_{k=1}^{K} \bar{m}_{\ell ok} + o(\varepsilon) \\
\mu_{\ell n} &= 1 - S - \mu_{hn} = y - \varepsilon \sum_{k=1}^{K} \bar{m}_{\ell ok} + o(\varepsilon) \\
\mu_{\ell ok} &\equiv \varepsilon m_{\ell ok} = \varepsilon \bar{m}_{\ell ok} + o(\varepsilon) \\
\mu_{hok} &= s_k - \varepsilon \bar{m}_{\ell ok} + o(\varepsilon).
\end{aligned}$$

The trading specialist allocation is $\nu_k = \bar{\nu}_k + o(1)$, where

$$\bar{\nu}_k = \frac{(1-q)x}{q\bar{\omega}} \frac{\bar{m}_{\ell ok}}{\bar{\mu}_{hn}} = \frac{\bar{m}_{\ell ok}}{\sum_{j=1}^K \bar{m}_{\ell oj}}, \tag{70}$$

where the first inequality follows from equation (65) when $\varepsilon = 0$, and the second inequality follows from the restriction that $\sum_{k=1}^K \nu_k = 1$. The instantaneous rate of contact between buyers and sellers is

$$M_k = \gamma_d s_k - \varepsilon \bar{m}_{\ell ok} + o(\varepsilon).$$

The indifference condition (62) shows that surplus of a outright sale of asset k is

$$\Sigma_k = \varepsilon \frac{\bar{\omega}\bar{\nu}_k\bar{\mu}_{hn}}{M_k(1-q)} + o(\varepsilon) = \varepsilon \frac{x}{q} \frac{\bar{m}_{\ell ok}}{\gamma_d s_k} + o(\varepsilon), \tag{71}$$

where the second equality follows from plugging in the expression for $\bar{\nu}_k\bar{\mu}_{hn}$ as well as the first-order approximation of M_k. Note that the surplus of a transaction is indeed positive for small enough ε.

Step 4: No-swap condition with a general matching function. To complete the existence proof we need to verify the no-swap condition. That is, one needs to verify that a seller of asset k prefers to search for an outright sale with a buyer rather than for a swap with a high-utility owner of some other asset.

We derive a sufficient condition. Consider an individual ℓok investor who seeks to swap his asset with some hoj investor. In our candidate equilibrium, no other ℓok investor searches for such a swap. Let $n_{kj} = 0$ be the intensity with which ℓok investors search for hoj investors, and ν_{jk} the intensity with which hoj investors search for ℓok investors. Evidently, the total rate of contact between ℓok and hoj investors is

$$M(2\lambda n_{kj}\mu_{\ell ok}, \nu_{jk}\mu_{hoj}) = 0,$$

since $n_{kj} = 0$. However, the Poisson intensity with which an individual ℓok investors establishes contact with hoj investors is not well defined as:

$$\frac{M(2\lambda n_{kj}\mu_{\ell ok}, \nu_{jk}\mu_{hoj})}{n_{kj}\mu_{\ell ok}} \text{“}=\text{”} \frac{0}{0}.$$

A natural way to address the problem of defining contact intensity when one side of the market does not search is to slightly "perturb" equilibrium trading strategies. Namely, we assume that a very small measure α of ℓok investors allocate all of their specialists to the search of a swap with hoj investors. As α goes to zero, ℓok investors establish contact with hoj investors with Poisson intensity:

$$\lim_{\alpha \to 0} \frac{M(2\lambda\alpha, \nu_{jk}\mu_{hoj})}{\alpha} = 2\lambda \frac{\partial M}{\partial a}(0, \nu_{jk}\mu_{hoj}) \leq 2\lambda \frac{\partial M}{\partial a}(0, s_j), \tag{72}$$

where ν_{jk} is the fraction of trading specialist that hoj investors allocate to the search of ℓok investors. The last inequality follows because $\nu_{jk} \leq 1$ and $\mu_{hoj} \leq s_j$, so $M(a, \nu_{jk}\mu_{hoj}) \leq M(a, s_j)$ for all $a \geq 0$.

Conditional on establishing a contact, the surplus of a transaction is $\Sigma_k - \Sigma_j$, and the bargaining power of the seller is assumed to be equal to q. Thus, a ℓok investor will not search for this swap if

$$2\lambda \frac{\partial M}{\partial a}(0, s_j) q (\Sigma_k - \Sigma_j) \leq \frac{M(2\lambda\mu_{\ell ok}, \nu_k\mu_{hn})}{\mu_{\ell ok}} q\Sigma_k, \tag{73}$$

that is, if the net utility of allocating a trading specialist to the swap, on the left-hand side, is less than the net utility of allocating the specialist to an outright sale, on the right-hand side. Recalling that $2\lambda = 1/\varepsilon$, $\mu_{\ell ok} = \varepsilon m_{\ell ok}$, and using the expansions derived above, the condition (73) can be simplified to

$$\begin{aligned}
& \frac{1}{2\varepsilon} \frac{\partial M}{\partial a}(0, s_j) q (\Sigma_k - \Sigma_j) \leq \frac{M(2\lambda\mu_{\ell ok}, \nu_k\mu_{hn})}{\varepsilon m_{\ell ok}} q\Sigma_k \\
\Leftrightarrow \quad & \frac{1}{2\varepsilon} \frac{\partial M}{\partial a}(0, s_j) q \frac{x}{q\gamma_d} \left(\varepsilon \frac{\bar{m}_{\ell ok}}{s_k} - \varepsilon \frac{\bar{m}_{\ell oj}}{s_j} + o(\varepsilon) \right) \leq \frac{\gamma_d s_k + o(1)}{\varepsilon(\bar{m}_{\ell ok} + o(\varepsilon))} q \frac{x}{q\gamma_d} \left(\varepsilon \frac{\bar{m}_{\ell ok}}{s_k} + o(\varepsilon) \right) \\
\Leftrightarrow \quad & \frac{\partial M}{\partial a}(0, s_j) \left(\frac{\bar{m}_{\ell ok}}{s_k} - \frac{\bar{m}_{\ell oj}}{s_j} \right) + o(1) \leq \gamma_d + o(1).
\end{aligned}$$

Therefore, a sufficient condition for (73) to hold when ε is small enough is

$$\frac{\partial M}{\partial a}(0, s_j) \left(\frac{\bar{m}_{\ell ok}}{s_k} - \frac{\bar{m}_{\ell oj}}{s_j} \right) < \gamma_d. \tag{74}$$

If the assets all have the same number of tradeable shares $(s_1, \ldots, s_K) = \hat{s} = (S/K, \ldots, S/K)$, then $\bar{m}_{\ell ok} = \bar{m}_{\ell oj}$, and (74) holds. Because $(\bar{m}_{\ell o1}, \ldots, \bar{m}_{\ell oK})$ are continuous function of $(s_1, \ldots, s_K)$, we conclude that there is a neighborhood $N \subset \mathbb{R}_+^K$ of $\hat{s}$ such that (74) holds for all $s \in N$. Proposition 6 then follows.

Step 5: No-swap condition with the bilinear matching function. With a bilinear matching function $M(a, b) = ab$, the no-swap condition has a simple form. This follows from the observation that, when $\varepsilon = 0$,

$$M(\bar{m}_{\ell ok}, \bar{\nu}_k \bar{\mu}_{hn}) = \gamma_d s_k,$$

where $\bar{\mu}_{hn} = y - S$ and

$$\bar{\nu}_k = \frac{\bar{m}_{\ell ok}}{(\sum_{j=1}^K \bar{m}_{\ell oj})}.$$

Using $M(a, b) = ab$, we obtain

$$\frac{\bar{m}_{\ell ok}^2 (y - S)}{\sum_{j=1}^K \bar{m}_{\ell oj}} = \gamma_d s_k, \tag{75}$$

Taking square roots on both sides and summing over $k \in \{1, \ldots, K\}$, we obtain

$$\sum_{k=1}^K \bar{m}_{\ell ok} = \frac{\gamma_d}{y - S} \left(\sum_{k=1}^K \sqrt{s_k} \right)^2.$$

Plugging this back in equation (75), we find:

$$\bar{m}_{\ell ok} = \frac{\gamma_d \sum_{j=1}^K \sqrt{s_j}}{y - S} \sqrt{s_k}.$$

Plugging this into the no-swap condition (74), noting that $\partial M / \partial a(0, s_j) = s_j$, and simplifying, we obtain

$$s_j \left(\frac{1}{\sqrt{s_k}} - \frac{1}{\sqrt{s_j}} \right) < B,$$

where

$$B \equiv \frac{y - S}{\sum_{j=1}^K \sqrt{s_j}}.$$

Rearranging this last inequality we find that, for all $s_k < s_j$,

$$\sqrt{s_k} > \sqrt{s_j} \frac{\sqrt{s_j}}{\sqrt{s_j} + B}.$$

This condition holds if and only if it holds when $s_k = \min\{s_i\}$ and $s_j = \max\{s_i\}$. Rearranging this expression yields to the condition of Proposition 4.

A.3.3 Proof of Propositions 8 and 9

First recall that, when $\varepsilon = 0$, $M_k = M(\bar{m}_{\ell ok}, \bar{\nu}_k \bar{\mu}_{hn}) = \gamma_d s_k$. Equation (70) shows that $\bar{\nu}_k = C\bar{m}_{\ell ok}$, for some constant $C > 0$. Hence,

$$M_k = M(\bar{m}_{\ell ok}, C\bar{m}_{\ell ok}) = \gamma_d s_k.$$

Because the function M is increasing in both arguments, it immediately follows that $\bar{m}_{\ell ok}$ and thus $\bar{\nu}_k$ can both be viewed as strictly increasing functions of s_k. That is, $s_k > s_j$ implies both $\bar{m}_{\ell ok} > \bar{m}_{\ell oj}$ and $\bar{\nu}_k > \bar{\nu}_j$. Because equilibrium objects are continuous functions of ε, these strict inequalities also hold when ε is close enough to zero. Also, for $s_k > s_j$, we note that

$$\frac{M(\bar{m}_{\ell ok}, \bar{\nu}_k \bar{\mu}_{hn})}{\bar{m}_{\ell ok}} \quad = \quad \frac{M(\bar{m}_{\ell ok}, C\bar{m}_{\ell ok})}{\bar{m}_{\ell ok}} = \frac{1}{\psi}\frac{M(\psi\bar{m}_{\ell oj}, \psi C\bar{m}_{\ell oj})}{\bar{m}_{\ell oj}} > \frac{M(\bar{m}_{\ell oj}, C\bar{m}_{\ell oj})}{\bar{m}_{\ell oj}},$$

if the matching function has increasing returns to scale, and where $\psi = \bar{m}_{\ell ok}/\bar{m}_{\ell oj} > 1$. Evidently, the reverse inequality holds if the matching function has decreasing returns to scale. By continuity, we find that, if the matching function has increasing returns to scale, then s long as ε is close enough to zero, $s_k > s_j$ implies that $M_k/m_{\ell ok} > M_j/m_{\ell oj}$. Now recall that

$$\Sigma_k = \frac{\varepsilon\omega}{1-q}\frac{\nu_k \mu_{hn}}{M_k} = \frac{\varepsilon\omega\mu_{hn}C}{1-q}\frac{m_{\ell ok}}{M_k}.$$

Thus, if the matching function has increasing returns to scale, then $s_k > s_j$ implies that $\Sigma_k < \Sigma_j$, and vice-versa if it has decreasing returns to scale. Now equation (29) shows that

$$rp_k = \delta - \varepsilon\omega - (\gamma + r(1-q))\,\Sigma_k,$$

so that $s_k > s_j$ implies that $p_k > p_j$.

ow, the dollar turnover is

$$\begin{aligned}
\frac{p_k M_k}{p_k s_k} &= \frac{M_k}{s_k} \\
&= \frac{\gamma_d s_k - \varepsilon\bar{m}_{\ell ok} + o(\varepsilon)}{s_k} \\
&= \gamma_d - \frac{\bar{m}_{\ell ok}}{s_k} + o(\varepsilon) \\
&= \gamma_d - \gamma_d \frac{\bar{m}_{\ell ok}}{M(\bar{m}_{\ell ok}, \omega\bar{m}_{\ell ok})} + o(\varepsilon),
\end{aligned}$$

where ω is some constant of proportionality. The last line follows because, when $\varepsilon = 0$, $M_k = \gamma_d s_k$. Hence turnover increases in the ration $M_k/\bar{m}_{\ell ok}$ and the result of Proposition 8 follows.

Lastly, note that dollar trading volume is

$$p_k M_k = \left(\frac{\delta}{r} + o(1)\right)(\gamma_d s_k + o(1)) = (\gamma_d \delta/r)\, s_k + o(1),$$

which proves Proposition 9.

A.4 Proof of Propositions 10 and 11

Consider an economy with two assets $k \in \{1, 2\}$. We conjecture an equilibrium where $\ell o1$ and hn investors search for each others in order to conduct an outright sale of asset 1, and $\ell o2$ and $ho1$ investors search for each others in order to swap asset 1 for asset 2. Thus, a $\ell o2$ investor end up selling his asset by conducting to consecutive transactions: he first swap asset 2 in exchange for asset 1 with some $ho2$ investor, and then sells asset 1 to some hn investor.

Step 1: steady-state distribution of types. Adopting the notation $m_{\ell o1} = 2\lambda\mu_{\ell o1}$ with $\varepsilon = 1/(2\lambda)$, we find that the steady-state distribution of types solve the system of inflow-outflow equations:

$$\begin{aligned}
\gamma_u \mu_{\ell n} &= \gamma_d \mu_{hn} + m_{\ell o1}\mu_{hn} \\
\gamma_d \mu_{ho1} + \mu_{ho1} m_{\ell o2} &= m_{\ell o1}\mu_{hn} + \gamma_u \varepsilon m_{\ell o1} \\
\gamma_u \varepsilon m_{\ell o2} + \mu_{ho1} m_{\ell o2} &= \gamma_d \mu_{ho2}.
\end{aligned}$$

The first equation is for hn investors, the second equation for $\ell o1$ investors, and the third for $ho2$ investors. As usual, we also have

$$\begin{aligned}
\mu_{hok} + \varepsilon m_{\ell ok} &= s_k \\
\sum_{k=1}^{2} (\mu_{hok} + \varepsilon m_{\ell ok}) + \mu_{hn} + \mu_{\ell n} &= 1,
\end{aligned}$$

for $k \in \{1, 2\}$. Substituting $\mu_{hok} = s_k - \varepsilon m_{\ell ok}$, and $\mu_{\ell n} = 1 - S - \mu_{hn}$ into the above inflow-outflow equations, we obtain the system

$$\gamma_u(1 - S) = \gamma\mu_{hn} + m_{\ell o1}\mu_{hn} \tag{76}$$

$$\gamma_d s_1 + (s_1 - \varepsilon m_{\ell o1})\, m_{\ell o2} = \mu_{hn} m_{\ell o1} + \gamma\varepsilon m_{\ell o1} \tag{77}$$

$$\gamma_d s_2 = m_{\ell o2}(s_1 - \varepsilon m_{\ell o1}) + \gamma\varepsilon m_{\ell o2}, \tag{78}$$

where $\gamma \equiv \gamma_u + \gamma_d$. Equation (78) shows that

$$m_{\ell o2} = \frac{\gamma_d s_2}{s_1 - \varepsilon m_{\ell o1} + \gamma\varepsilon}, \tag{79}$$

while adding equations (77) and (78) implies

$$\gamma_d S = \mu_{hn} m_{\ell o1} + \gamma\varepsilon\,(m_{\ell o1} + m_{\ell o2})\,. \tag{80}$$

Let $m_{\ell o} = m_{\ell o1} + m_{\ell o2}$ and recall that

$$\mu_{hn} = y - (\mu_{ho1} + \mu_{ho2}) = y - S + \varepsilon m_{\ell o},$$

where $y \equiv \gamma_u/(\gamma_u + \gamma_d)$. Plugging this back into (80), we find that

$$m_{\ell o1} = \frac{\gamma_d S - \gamma\varepsilon m_{\ell o}}{y - S + \varepsilon m_{\ell o}} \equiv F(m_{\ell o}, \varepsilon),$$

for some function $F(m, \varepsilon)$ that is strictly decreasing in both its arguments. Thus, one obtain the one equation in one unknown problem

$$m_{\ell o} = m_{\ell o1} + m_{\ell o2} = F(m_{\ell o}, \varepsilon) + \frac{\gamma_d s_2}{s_1 + \gamma\varepsilon - \varepsilon F(m_{\ell o}, \varepsilon)},$$

where the second term on the right-hand side is obtained by plugging in $m_{\ell o1} = F(m_{\ell o}, \varepsilon)$ into equation (79). Given a solution to this equation, one can calculate the steady-state distribution of types using the other equations.

When $\varepsilon = 0$, this equation reduces to

$$m_{\ell o} = \bar{m}_{\ell o} = \frac{\gamma_d S}{\frac{\gamma_u}{\gamma_u + \gamma_d} - S} + \frac{\gamma_d s_2}{s_1}.$$

Since $m \mapsto m - F(m, 0)$ is strictly increasing, an application of the implicity function theorem shows that a solution exists for ε close to zero, and that $m_{\ell o} \to \bar{m}_{\ell o}$ as ε goes to zero. This implies that, for ε close to zero, the distribution of types admits the first-order approximation

$$\begin{aligned}
\mu_{\ell o1} &= \varepsilon \bar{m}_{\ell o1} + o(\varepsilon) \\
\mu_{ho1} &= s_1 - \varepsilon \bar{m}_{\ell o1} + o(\varepsilon) \\
\mu_{\ell o2} &= \varepsilon \bar{m}_{\ell o2} + o(\varepsilon) \\
\mu_{ho2} &= s_2 - \varepsilon \bar{m}_{\ell o2} + o(\varepsilon) \\
\mu_{hn} &= y - S + \varepsilon \left(\bar{m}_{\ell o1} + \bar{m}_{\ell o2}\right) + o(\varepsilon) \\
\mu_{\ell n} &= 1 - y - \varepsilon \left(\bar{m}_{\ell o1} + \bar{m}_{\ell o2}\right) + o(\varepsilon).
\end{aligned}$$

Step 2: Bellman Equations. We now turn to the Bellman equations. Adopting the notation $\varepsilon \sigma_k \equiv V_{hok} - V_{hn} + V_{\ell n} - V_{\ell ok}$, the continuation utilities solve the system:

$$\begin{aligned}
rV_{hn} &= \gamma_d \left(V_{\ell n} - V_{hn}\right) + m_{\ell o1}(1-q)\varepsilon \sigma_1 \\
rV_{ho1} &= \delta + \gamma_d \left(V_{\ell o1} - V_{ho1}\right) + m_{\ell o2}(1-q)\varepsilon \left(\sigma_2 - \sigma_1\right) \\
rV_{\ell o1} &= \delta - x + \gamma_u (V_{ho1} - V_{\ell o1}) + \frac{\mu_{hn}}{\varepsilon} q \varepsilon \sigma_1 \\
rV_{ho2} &= \delta + \gamma_d (V_{\ell o2} - V_{ho2}) \\
rV_{\ell o2} &= \delta - x + \gamma_u (V_{ho2} - V_{\ell o2}) + \frac{\mu_{ho1}}{\varepsilon} q \varepsilon \left(\sigma_2 - \sigma_1\right) \\
rV_{\ell n} &= \gamma_u \left(V_{hn} - V_{\ell n}\right).
\end{aligned}$$

Combining these equations we find

$$\begin{aligned}
\left[q\mu_{hn} + \varepsilon \left(r + \gamma + m_{\ell o}(1-q)\right)\right] \sigma_1 &= x + \varepsilon m_{\ell o2}(1-q)\sigma_2 \\
\left[q\mu_{ho1} + \varepsilon(r+\gamma)\right] \sigma_2 &= x + \left[q\mu_{ho1} - \varepsilon m_{\ell o1}(1-q)\right] \sigma_1.
\end{aligned}$$

Thus $\sigma' \equiv (\sigma_1, \sigma_2)$ solves a system of linear equations of the form

$$A(\varepsilon)\sigma = \begin{bmatrix} x \\ x \end{bmatrix},$$

for some matrix A whose coefficient are continuously differentiable in ε. When $\varepsilon = 0$

$$A(0) = \begin{bmatrix} q\bar{\mu}_{hn} & 0 \\ -qs_1 & qs_1 \end{bmatrix}.$$

When $\varepsilon = 0$, the solution is

$$\begin{aligned}
\bar{\sigma}_1 &= \frac{x}{q\bar{\mu}_{hn}} \\
\bar{\sigma}_2 &= \bar{\sigma}_1 + \frac{x}{qs_1}.
\end{aligned}$$

Because $A(0)$ is invertible, an application of the Implicity Function Theorem shows that a unique solution exists for ε close to zero.

Step 3: verification. We need to verify that an individual investor picks an optimal allocation of trading specialist, taking as given the allocation chosen by all other investors.

For a hn investor, the net utility of allocating a specialist to the search of $\ell o1$ investors is $(1-q)m_{\ell o1}\varepsilon\sigma_1 > 0$. The net utility of allocating specialists to the search of $\ell o2$ investors is zero, because $\ell o2$ do not search for an outright sale. The net utility of searching for other investors is non-positive. Thus, a hn investor finds it optimal to allocation all of his trading specialist to the search of $\ell o1$ investors.

For a $\ell o1$ investor, the net utility of allocating a specialist to the search of hn investor is $q\mu_{hn}\sigma_1 > 0$. The net utility of searching for a swap with $ho2$ investorsis proportional to $\sigma_1 - \sigma_2 < 0$. The net utility of searching for other investors is non-positive. Thus, a $\ell o1$ investor finds it optimal to allocation all of his trading specialist to the search of hn investors.

For a $\ell o2$ investor, the net utility of allocating specialists to the search of a swap with $ho1$ investor is proportional to $\sigma_2 - \sigma_1 > 0$. The net utility of allocating specialists to the search of hn investors is zero, because hn investors do not search for an outright purchase of asset 2. The net utility of searching for other investors is non-positive. Thus, a $\ell o2$ investor finds it optimal to allocation all of his trading specialist to the search of $ho1$ investors.

For a $ho1$ investor, the net utility of searching for a $\ell o2$ investor is proportional to $\sigma_2 - \sigma_1$. The net utility of searching for other investors is non-positive. Thus, a $ho1$ investor finds it optimal to allocation all of his trading specialist to the search of $\ell o2$ investors.

For a $ho2$ investor, the net utility of searching for a swap with $\ell o1$ investor is proportional to $\sigma_1 - \sigma_2 < 0$, and the net utility of searching for other investors is non-positive. Thus, not searching is optimal for a $ho2$ investor.

Lastly, for a ℓn investor the utility of searching for other investors is also non-positive, so that not searching is also optimal.

Pricing. The price of asset 1 is

$$\begin{aligned} rp_1 &= r\Delta V_{h1} - r(1-q)\varepsilon\sigma_1 \\ &= \delta - \left[\gamma_d + r(1-q) + m_{\ell o1}(1-q)\right]\varepsilon\sigma_1 + m_{\ell o2}\varepsilon\left(\sigma_2 - \sigma_1\right) \\ &= \delta - \varepsilon\left[\gamma_d + (1-q)\left(r + \bar{m}_{\ell o1}\right)\right]\bar{\sigma}_1 - \varepsilon\frac{\gamma_d s_2}{s_1}\frac{x}{q s_1} + o(\varepsilon) \\ &= \delta - \varepsilon A_1(S) + \varepsilon B_1 \frac{s_2}{s_1^2} + o(\varepsilon), \end{aligned}$$

for some positive function $A_1(S)$ of S and some positive constant B_1. The last line follows from noting that $\bar{m}_{\ell o1}$ and $\bar{\sigma}_1$ only depend on the total number of tradeable share, S. Similarly, the price of asset 2 is

$$\begin{aligned} rp_2 &= r\Delta V_{h2} - r(1-q)\varepsilon\sigma_2 \\ &= \delta - \gamma_d\varepsilon\sigma_2 - m_{\ell o1}(1-q)\sigma_1 - r(1-q)\varepsilon\sigma_2 \\ &= \delta - \gamma_d\varepsilon\sigma_1 - m_{\ell o1}(1-q)\sigma_1 - r(1-q)\varepsilon\sigma_1 - \varepsilon\left(\gamma_d + r(1-q)\right)\left(\sigma_2 - \sigma_1\right) \\ &= \delta - \varepsilon A_2(S) - B_2\frac{x}{q s_1} + o(\varepsilon), \end{aligned}$$

for some positive function $A_2(S)$ of S and some positive constant B_2. The comparative statics of the Proposition then follows.

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