Liquidity Premia in Dynamic Bargaining Markets

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Motivation

- Controlling for risk premia, expected stock return is
  - positively related to bid-ask spread.
  - negatively related to turnover and trading volume.

Motivation

- A search model to explain the cross-sectional distribution of stock returns
  - abstracting from risk premia
  - focusing exclusively on liquidity

The Economy

- Time: continuous, $t \in [0, \infty)$
- $[0, 1]$ of risk-neutral, infinitely lived agents
- K assets in positive supply

Agents

- Agents:
  - consume non-storable numraire good called "cash"
  - have a technology to produce cash at unit marginal cost
  - can only hold 0 or 1 share of only one asset

Agents

- 2-types of agents
  - low-valuation or high valuation for holding assets
    - high valuation, (per unit) utility flow δ
    - low valuation, (per unit) utility flow $\delta - x$
  - Random switch from low(high) to high(low) with intensity $\gamma_u(\gamma_d)$

Agents

- Set of agent types

$$I = \{hn, ln, ho1, \cdots, hoK, lo1, \cdots, loK\}$$

  - hn, "buyers"
  - lok, "seller" of asset k
  - μ_i, fraction of agents of type $i \in I$

Assets

- K assets in positive supply
  - dividend rate of δ
  - $s_k \in (0, 1)$, measure of tradable share
  - $s \equiv (s_1, \cdots, s_K)$, distribution of tradable shares
  - $\sum_{k=1}^{K} s_k \equiv S < \frac{\gamma_u}{\gamma_u + \gamma_d} \Rightarrow$ total asset supply smaller than the steady-state measure of high valuation agents.

Matching Technology

- No centralized market
- Agents are endowed with a mass $\bar{\nu} = 1$ of trading specialists
- Trading specialist of type (i, j)
  - works for a agent of type i
  - specializes in contacting specialists working for investors of type j
- An agent of type i maintains ν_{ij} specialist of type (i, j) s.t.

$$\sum_j \nu_{ij} \leq \bar{\nu} = 1$$

Matching Technology

- Trading specialist matched randomly at Poisson arrival times with intensity λ
- Contacts between specialists of type (i, j) and (j, i), for $i \neq j$, occurs continually at a total rate of

$$\underbrace{\mu_i \nu_{ij}}_{mass\ of\ (i,j)\ spec} \underbrace{\lambda}_{cnt\ rate} \underbrace{\mu_j \nu_{ji}}_{mass\ of\ (j.i)\ spec} + \mu_j \nu_{ji} \lambda \mu_i \nu_{ij} = 2\lambda \mu_i \nu_{ij} \mu_j \nu_{ji}$$

The Equilibrium

Steady-state, symmetric Nash equilibrium:

- prices, $p = (p_1, \cdots, p_K)$
- continuation utilities $V = (V_{hn}, V_{hok}, V_{lok}, V_{\ln})_{1 \leq k \leq K}$
- distribution of types, $\mu = (\mu_{hn}, \mu_{hok}, \mu_{lok}, \mu_{\ln})_{1 \leq k \leq K}$
- trading specialists allocation, $\nu \gg 0$

that satisfy

- steady-state
- optimality
- no-swap and pricing

Steady State Conditions

- Steady-state distribution of types:

$$
\begin{aligned}
s_k &= \mu_{lok} + \mu_{hok} && (1)\\
1 &= \sum_{k=1}^{k} (\mu_{lok} + \mu_{hok}) + \mu_{hn} + \mu_{ln} && (2)\\
\gamma_u \mu_{ln} &= \gamma_d \mu_{hn} + \sum_{k=1}^{K} 2\lambda \nu_k \mu_{hn} \mu_{lok} && (3)\\
\gamma_d \mu_{hok} &= \gamma_u \mu_{lok} + 2\lambda \nu_k \mu_{hn} \mu_{lok}, \textit{for } k \in \{1, \cdots, K\} && (4)
\end{aligned}
$$

Optimality Conditions

- Bellman Equations

$$
\begin{aligned}
rV_{hn} &= \max_{\tilde{\nu}_1,\cdots,\tilde{\nu}_k} \{\gamma_d(V_{ln} - V_{hn}) \\
&\quad + \sum_{k=1}^{K} 2\lambda\tilde{\nu}_k\mu_{lok}(V_{hok} - V_{hn} - p_k)\} && (5) \\
rV_{hok} &= \delta + \gamma_d(V_{lok} - V_{hok}) && (6) \\
rV_{lok} &= \delta - x + \gamma_u(V_{hok} - V_{lok}) + 2\lambda\nu_k\mu_{hn}(V_{ln} - V_{lok} + p_k) && (7) \\
rV_{ln} &= \gamma_u(V_{hn} - V_{ln}) && (8)
\end{aligned}
$$

No-Swap Condition

- No-swap condition

$$2\lambda\nu_{jk}\mu_{hoj}q(V_{loj} - V_{lok} + V_{hok} - V_{hoj})$$

$$< 2\lambda\nu_{k}\mu_{hn}q(V_{hok} - V_{hn} + V_{ln} - V_{lok}) \qquad (9)$$

Pricing Condition

- Pricing condition
  lok agent receives a fixed fraction $q \in (0,1)$ of total surplus of the transaction between *hn* and *lok* which is

$$(V_{hok} - V_{hn}) - (V_{lok} - V_{ln}) \equiv \Delta V_{hk} - \Delta V_{lk}$$

in equilibrium

$$p_k = q\Delta V_{hk} + (1-q)\Delta V_{lk} \tag{10}$$

Search Indifference

- Since (5) is a linear program, $\nu \gg 0$ implies that hn investors are indifferent between searching for any two assets.
- By F.O.C. of (5) and (10), we have

$$2\lambda\mu_{lok}(V_{hok} - V_{hn} - p_k) = 2\lambda\mu_{loj}(V_{hoj} - V_{hn} - p_j)$$

$$\Leftrightarrow \mu_{lok}(\Delta V_{hk} - \Delta V_{lk}) = \mu_{loj}(\Delta V_{hj} - \Delta V_{lj})$$

- The above equation suggests a negative relationship between liquidity and bid-ask spread

Existence

- Proposition 1:

  Given an allocation v of trading specialist, the system (1)-(4) has a unique solution

$$\mu = (\mu_{hn}, \mu_{hok}, \mu_{lok}, \mu_{\ln})_{1 \leq k \leq K} \in [0,1]^{2K+2}$$

Existence

- Proposition 2 (Uniqueness):

  There is at most one symmetrice quilibriium.

Existence

- Proposition 3 (Existence with almost-identical Assests):

  Let $\hat{s} = (S/K, \cdots S/K)$. Then, there is a neighborhood $N \subset R_+^K$ of $\hat{s}$, such that, for all $s \in N$, there is a symmetric equilibrium.

Existence

- Proposition 4 (Existence in an almost-frictionless Market):

Suppose that the distribution $(s_1, \cdots, s_K)$ of tradeable shares satisfies the no-swap condition

$$\sqrt{\max\{s_j\}} - \sqrt{\max\{s_k\}} < \sqrt{\max\{s_j\}} \frac{B}{B + \sqrt{\max\{s_j\}}}$$

where $B = (\frac{\gamma_u}{\gamma_u + \gamma_d} - S)(\sum_{i=1}^{K} \sqrt{s_i})^{-1}$

Then there exists some $\bar{\lambda} \in R_+$ such that a symmetric equilibrium exists for all $\lambda > \bar{\lambda}$.

Liquidity-Returns Relationships

- In equilibrium, cross-sectional variation in tradeable shares creates a negative relationship between asset returns and liquidity
- The equilibrium cross-sectional variation in asset returns arises endogenously and is explained by the distribution $s = (s_1, \cdots, s_K)$ of tradeable shares

Three Equations

- The analysis is based on three equations.
- The first one is derived from the pricing equation (10):

$$p_k = \frac{\delta}{r} - \frac{W}{r} - (1 + \frac{\gamma_d}{r(1-q)})\frac{W}{(2\lambda\mu_{lok})} \qquad (11)$$

where W is a constant from the search indifference condition

- An asset that is easier to find (has larger μ_{lok}) is sold at a higher price

Three Equations

- The second equation is derived from the indifference condition

$$As_k \frac{1}{\mu_{lok}^2} + B\frac{1}{\mu_{lok}} + C = \frac{1}{W} \qquad (12)$$

for some positive constants A, B, and C, which do not depend on k

- This equation relates the measure μ_{lok} of sellers to the measure s_k of tradeable shares

Three Equations

- The third equation is

$$\frac{\mu_{lok}}{s_k} = \frac{\gamma_d}{\gamma_d + \gamma_u + 2\lambda v_k \mu_{hn}} \qquad (13)$$

- This equation relates the allocation v_k of trading specialists to the measure μ_{lok} of sellers and the measure s_k of tradeable shares

Liquidity-Returns Relationships

- Proposition 5:

In equilibrium, $s_k > s_j$ implies that $\mu_{lok} > \mu_{loj}$, $v_k > v_l$, $p_k > p_j$, $R_k < R_j$, and $\Delta V_{hk} - \Delta V_{lk} < \Delta V_{hj} - \Delta V_{lj}$.

Proof of Proposition 5

- Based on the equation (12), if $s_k > s_j$, we have $\mu_{lok} > \mu_{loj}$
- If $\mu_{lok} > \mu_{loj}$,we have $v_k > v_l$
  - If $\mu_{lok} > \mu_{loj}$,from (12), we have $\frac{s_k}{\mu_{lok}} > \frac{s_j}{\mu_{loj}}$
  - If $\frac{s_k}{\mu_{lok}} > \frac{s_j}{\mu_{loj}}$,from (13), we have $v_k > v_l$

Proof of Proposition 5

- From equation (11), if $\mu_{lok} > \mu_{loj}$,we have $p_k > p_j$
- Let $R_k = \frac{\delta}{p_k}$, we have $R_k < R_j$
- From the search indifference condition, we have $\Delta V_{hk} - \Delta V_{lk} < \Delta V_{hj} - \Delta V_{lj}$

Empirical Evidence

- The model provides predictions regarding the relationship between liquidity and expected returns across industry portfolios as opposed to individual stocks.
- The studies that focus on individual stocks may still provide an indirect test of the model, as the liquidity level of an individual stock is, in part, explained by the liquidity level of its industry.

Empirical Evidence

- Some emperical studies suggest that assets with a larger number of tradeable shares are more liquid
- Chan, Chan, and Fong[2004] study the impact of a reduction in the number of tradeable shares on asset liquidity in the Hong Kong stock market
- Zhang, Tian, and Wirjanto[2007] document a negative relationship between tradeable shares and risk-adjusted cross-sectional returns in the China stock market.